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Algebraic solutions of the sixth Painlevé equation

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ABSTRACT

We describe all finite orbits of an action of the extended modular group $\bar{\Lambda}$ on conjugacy classes of $SL_2(\mathbb{C})$-triples. The result is used to classify all algebraic solutions of the general Painlevé VI equation up to parameter equivalence.

1. Introduction

Modular group $\Gamma = PSL_2(\mathbb{Z})$ consists of 2×2 matrices with integer entries and unit determinant, considered up to overall sign. It has a presentation $\Gamma = \langle s, t \mid s^3 = t^2 = 1 \rangle$, and is known to be isomorphic to the quotient of 3-braid group $\mathcal{B}_3$ by its center $\mathcal{Z} \cong \mathbb{Z}$. The kernel of the canonical homomorphism $\Gamma \to PSL_2(\mathbb{Z}_2) \cong S_3$ defines a congruence subgroup $\Lambda \subset \Gamma$, also known as $\Gamma(2)$:

$$\Lambda = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid a, d \text{ odd};\ b, c \text{ even} \right\} / \{\pm 1\}.$$

There are isomorphisms $\Lambda \cong \mathcal{P}_3/\mathcal{Z} \cong \mathcal{F}_2$, where $\mathcal{P}_3$ denotes the group of pure 3-braids and $\mathcal{F}_2$ is the free group with 2 generators.

Extended modular groups $\bar{\Gamma}$ and $\bar{\Lambda}$ are obtained by replacing the unit determinant condition with $ad - bc = \pm 1$. These groups have the following presentations:

$$\bar{\Gamma} = \langle r, s, t \mid r^2 = s^3 = t^2 = (tr)^2 = (sr)^2 = 1 \rangle, \tag{1}$$

$$\bar{\Lambda} = \langle x, y, z \mid x^2 = y^2 = z^2 = 1 \rangle \cong C_2 * C_2 * C_2, \tag{2}$$

where

$$t = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad s = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \qquad r = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$x = rsts = \begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix}, \qquad y = rt = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad z = stsr = \begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix}.$$

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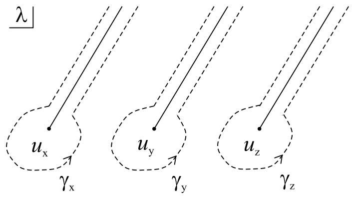


Fig. 1. Branch cuts and loops $\gamma_{x,y,z}$.

Note that Λ is isomorphic to the subgroup (of index 2) of $\bar{\Lambda}$ containing words of even length in x, y, z. Hence, given a $\bar{\Lambda}$ action on a set U and a point $u \in U$, the orbits $\bar{\Lambda}(u)$ and $\Lambda(u)$ are simultaneously finite or infinite.

In this paper the last observation is used to classify algebraic solutions of the sixth Painlevé equation (see [1]):

$$\frac{d^2w}{dt^2} = \frac{1}{2}\left(\frac{1}{w}+\frac{1}{w-1}+\frac{1}{w-t}\right)\left(\frac{dw}{dt}\right)^2 - \left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{w-t}\right)\frac{dw}{dt} + \frac{w(w-1)(w-t)}{2t^2(t-1)^2}\left((\theta_\infty-1)^2 - \frac{\theta_x^2 t}{w^2} + \frac{\theta_y^2(t-1)}{(w-1)^2} + \frac{(1-\theta_z^2)t(t-1)}{(w-t)^2}\right). \tag{PVI}$$

This is the most general ODE of the form $w'' = F(t, w, w')$, with F rational in w, w' and t, whose general solution has no movable branch points and essential singularities. It can therefore be analytically continued to a meromorphic function on the universal covering of $\mathbb{P}^1\backslash\{0, 1, \infty\}$.

A result from Watanabe [2] suggests that, roughly speaking, any solution of PVI is either (a) algebraic or (b) solves a Riccati equation or (c) cannot be expressed via classical functions. Known examples of algebraic solutions [3] turn out to be related to various mathematical structures, including e.g. Frobenius manifolds [4], symmetry groups of regular polyhedra [5,6], complex reflections [7], Grothendieck's dessins d'enfants and their deformations [8–10]. A few families of non-classical solutions have also been constructed in terms of Fredholm determinants, see [11,12].

In the case $\theta_x = \theta_y = \theta_z = 0$ a full classification of algebraic solutions has been obtained by Dubrovin and Mazzocco [5]. Their approach, followed to some extent in the present work, is based on the description of PVI as the equation of monodromy preserving deformation of Fuchsian systems of the form

$$\frac{d\Phi}{d\lambda} = \left(\frac{A_x}{\lambda-u_x}+\frac{A_y}{\lambda-u_y}+\frac{A_z}{\lambda-u_z}\right)\Phi, \quad A_\nu \in \mathfrak{sl}_2(\mathbb{C}), \tag{3}$$

where the poles u_ν are pairwise distinct, A_ν are 2×2 matrices independent of λ with eigenvalues $\pm\theta_\nu/2$ and

$$A_x + A_y + A_z = \begin{pmatrix} -\theta_\infty/2 & 0 \\ 0 & \theta_\infty/2 \end{pmatrix}, \quad \theta_\infty \neq 0.$$

The fundamental matrix $\Phi(\lambda)$ is a multivalued analytic function on $\mathbb{C}\backslash\{u_x, u_y, u_z\}$. Fix a basis of loops and branch cuts in $\pi_1(\mathbb{P}^1\backslash\{u_x, u_y, u_z, \infty\}, \infty)$ as shown in Fig. 1. To each branch of a solution of the PVI equation corresponds a unique (up to conjugation) triple of monodromy matrices $(M_x, M_y, M_z) \in G^3$, $G = SL_2(\mathbb{C})$ of $\Phi(\lambda)$ w.r.t. the loops $\gamma_x, \gamma_y, \gamma_z$. One consequence of isomonodromy is that analytic continuation of solutions of PVI induces an action of the pure braid group on 3 strings on the space of conjugacy classes of such triples (i.e. on the quotient $\mathcal{M} = G^3/G$ of three copies of G by diagonal conjugation by G). It extends to the standard Hurwitz action of the braid group $\mathcal{B}_3 = \langle \beta_x, \beta_z \mid \beta_x\beta_z\beta_x = \beta_z\beta_x\beta_z\rangle$ on G^3. Explicitly,

$$\beta_x : (M_x, M_y, M_z) \mapsto (M_x, M_z, M_zM_yM_z^{-1}),$$
$$\beta_z : (M_x, M_y, M_z) \mapsto (M_y, M_yM_xM_y^{-1}, M_z).$$

Observe that $\beta_z\beta_x$ acts on a representative triple $(M_x, M_y, M_z) \in \mathcal{M}$ by a cyclic permutation. The center $\mathcal{Z}$ of $\mathcal{B}_3$ is generated by $(\beta_z\beta_x)^3$ and therefore it acts on $\mathcal{M}$ trivially. This leads to an action of the modular group $\Gamma \cong \mathcal{B}_3/\mathcal{Z}$ on $\mathcal{M}$, with

$$s : (M_x, M_y, M_z) \mapsto (M_z, M_x, M_y), \tag{4}$$

$$t : (M_x, M_y, M_z) \mapsto (M_z, M_y, M_yM_xM_y^{-1}) \tag{5}$$

in the above notation. The action of $\bar{\Gamma}$ on $\mathcal{M}$ is obtained by adding the involution

$$r : (M_x, M_y, M_z) \mapsto (M_z^{-1}, M_y^{-1}, M_x^{-1}). \tag{6}$$

Lemma 1. *The transformations $s, t, r : \mathcal{M} \to \mathcal{M}$, as given by (4)–(6), satisfy the defining relations (1) of the extended modular group $\bar{\Gamma}$.*

As a corollary, we obtain the restriction of the $\bar{\Gamma}$ action to its level 2 subgroup $\bar{\Lambda}$:

Lemma 2. *The generators $x, y, z \in \bar{\Lambda}$ act on representative triples from $\mathcal{M}$ as follows:*

$$\begin{aligned} x &: \left(M_x, M_y, M_z\right) \mapsto \left(M_x^{-1}, M_y^{-1}, M_x M_z^{-1} M_x^{-1}\right), \\ y &: \left(M_x, M_y, M_z\right) \mapsto \left(M_y M_x^{-1} M_y^{-1}, M_y^{-1}, M_z^{-1}\right), \\ z &: \left(M_x, M_y, M_z\right) \mapsto \left(M_x^{-1}, M_z M_y^{-1} M_z^{-1}, M_z^{-1}\right). \end{aligned} \tag{7}$$

Proof. Both lemmas can be proved by direct calculation. □

Let us now describe the last action in more detail, introducing on $\mathcal{M}$ a suitable set of coordinates. Following [13], to a point $(M_x, M_y, M_z) \in \mathcal{M}$ we associate a 7-tuple $(p_x, p_y, p_z, p_\infty, X, Y, Z) \in \mathbb{C}^7$ given by

$$p_x = \operatorname{Tr} M_x, \qquad p_y = \operatorname{Tr} M_y, \qquad p_z = \operatorname{Tr} M_z, \qquad p_\infty = \operatorname{Tr}\left(M_z M_y M_x\right), \tag{8}$$

$$X = \operatorname{Tr}\left(M_y M_z\right), \qquad Y = \operatorname{Tr}\left(M_z M_x\right), \qquad Z = \operatorname{Tr}\left(M_x M_y\right). \tag{9}$$

Naive dimension of the quotient $\mathcal{M}$ is equal to 6 and thus it is not surprising that the above monodromy invariants are not all independent — there is a constraint

$$XYZ + X^2 + Y^2 + Z^2 - \omega_X X - \omega_Y Y - \omega_Z Z + \omega_4 = 4, \tag{10}$$

where

$$\omega_X = p_x p_\infty + p_y p_z, \qquad \omega_Y = p_y p_\infty + p_z p_x, \qquad \omega_Z = p_z p_\infty + p_x p_y, \tag{11}$$

$$\omega_4 = p_x^2 + p_y^2 + p_z^2 + p_\infty^2 + p_x p_y p_z p_\infty. \tag{12}$$

Remark 3. Boalch [7] refers to an equation equivalent to (10) as 'Fricke relation'. In the context of Painlevé VI, it was first obtained by Jimbo in [14], p.1140.

Remark 4. Four quantities (8) are related to PVI parameters by

$$p_\nu = 2\cos\pi\theta_\nu, \quad \nu = x, y, z, \infty. \tag{13}$$

Remaining three parameters X, Y, Z satisfying the Jimbo–Fricke relation (10) can be generically thought of as giving two PVI integration constants.

The $\bar{\Lambda}$ action (7) is defined for any group G. That $G = SL_2(\mathbb{C})$ in our case leads to important simplifications, in particular $\operatorname{Tr} M = \operatorname{Tr} M^{-1}$ for any $M \in G$. Monodromy parameters (8) are then fixed by the induced action of $\bar{\Lambda}$, and quadratic functions (9) transform according to the following:

Lemma 5. *The induced action of the generators $x, y, z \in \bar{\Lambda}$ on the parameters (9) is*

$$\begin{aligned} x(X, Y, Z) &= (\omega_X - X - YZ, Y, Z), \\ y(X, Y, Z) &= (X, \omega_Y - Y - ZX, Z), \\ z(X, Y, Z) &= (X, Y, \omega_Z - Z - XY). \end{aligned} \tag{14}$$

Proof. Using again that for $M \in SL_2(\mathbb{C})$ one has $\operatorname{Tr} M = \operatorname{Tr} M^{-1}$ and also $M + M^{-1} = \operatorname{Tr} M \cdot \mathbf{1}$ we find for example

$$\begin{aligned} x(X) &= \operatorname{Tr}\left(M_y^{-1} M_x M_z^{-1} M_x^{-1}\right) = \operatorname{Tr}\left(M_y M_x M_z M_x^{-1}\right) = p_x p_\infty - \operatorname{Tr}\left(M_y M_x M_z M_x\right) \\ &= p_x p_\infty - YZ + \operatorname{Tr}\left(M_y M_z^{-1}\right) = p_x p_\infty + p_y p_z - X - YZ. \end{aligned}$$

Proof of the other relations follows in a similar manner. □

Remark 6. After this work has been completed, we became aware of two recent papers [15,16], where the group $\bar{\Lambda}$ was introduced into Painlevé VI context in a way similar to ours and in particular its action (14) on monodromy invariants has been computed (cf. relations (2.10)–(2.12) in [15] and formula (37) in [16]). We also note another interesting recent preprint [17] on algebraic PVI solutions.

Idea of classification. Finite branch (in particular, algebraic) solutions of Painlevé VI necessarily lead to finite orbits of the $\mathcal{P}_3/\mathcal{Z} \cong \Lambda$ action on the space $\mathcal{M}$ of conjugacy classes of monodromy. Classification of such orbits is equivalent to finding all finite orbits of the action (7) of the extended modular group $\bar{\Lambda}$. Finally, the orbit $\bar{\Lambda}(m)$, $m \in \mathcal{M}$ can be finite only if the corresponding orbit of the induced $\bar{\Lambda}$ action (14) on $\mathbb{C}^3$ is finite.

Remark 7. One usually obtains explicit algebraic solution curves from monodromy by applying Jimbo's asymptotic formula [14] (or an appropriate modification of it) and computing sufficiently many terms in the Puiseux expansions of solutions near singular points. Another extremely useful tool, especially for solutions of high degree, are Kitaev's quadratic transformations [18,19].

In the next section, we classify all finite orbits of the action (14) (Theorem 1). It then turns out that the resulting 7-tuples of monodromy invariants completely determine Λ-orbits in $\mathcal{M}$ except in the case when $M_{x,y,z}$ can be simultaneously transformed into the upper triangular form. In Section 3, we give a complete (up to parameter equivalence) list of Painlevé VI solutions with finite branching. All of them are algebraic with one possible exception of Picard solutions; in that way our explicit computation confirms a recent result by Iwasaki [20].

Somewhat unexpectedly for the authors, the solutions corresponding to all possible finite Λ-orbits have already appeared in various papers [8,7,21–23,4,5,24,6,9,10]. However, four of them (solutions 13, 24, 43 and 44 below) were published with misprints, which are fixed in the present paper.

2. Finite orbits of $\bar{\Lambda}$

2.1. Orbit graphs

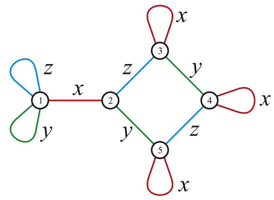
Our main subject in this section is the $\bar{\Lambda}$-action (14) which we consider as an action on $\mathbb{C}^3$ by fixing the parameters $\boldsymbol{\omega} = (\omega_X, \omega_Y, \omega_Z)$. To any orbit O of this action we associate a 3-colored (pseudo)graph $\Sigma(O)$ as follows:

- the vertices of $\Sigma(O)$ represent distinct points $\mathbf{r} = (X, Y, Z) \in O$,
- two vertices $a, b \in \Sigma(O)$ are connected by an undirected edge of color x, y or z if $x(a) = b$ (resp. $y(a) = b$ or $z(a) = b$),
- if a point $a \in \Sigma(O)$ is fixed by the transformation x, y or z, we assign to it a self-loop of the corresponding color.

In fact $\Sigma(O)$ is a Schreier coset graph as its vertices can be identified with the cosets of the stabilizer of any point in O. Also observe that the structure of (14) imposes a number of restrictions on $\Sigma(O)$, in particular it forbids multiple edges and simple cycles with only one edge of a given color.

Example 8. Set $\boldsymbol{\omega} = (0, 1, 1)$ and consider the orbit of the point $\mathbf{r} = (-1, 1, 1)$. It consists of 5 points with coordinates given below along with the orbit graph.

Point	X	Y	Z
1	-1	1	1
2	0	1	1
3	0	1	0
4	0	0	0
5	0	0	1



This orbit does not split under the action of non-extended modular group Λ. The same result is immediate for any $\bar{\Lambda}$-orbit whose graph contains at least one self-loop (recall that Λ consists of even-length words in x, y, z).

2.2. Symmetries

Before we move on to the classification, it is useful to look at the symmetries of the space of orbits and their relation to Bäcklund transformations for Painlevé VI.

Let $T : \mathcal{M} \to \mathcal{M}$ be an invertible map and let $\mathcal{O} \in \mathcal{M}$ be an orbit of the $\bar{\Lambda}$-action (7). If there exists an automorphism $\varphi \in \operatorname{Aut} \bar{\Lambda}$ compatible with T (i.e. $\lambda(T(u)) = T(\varphi(\lambda)(u))$ for any $\lambda \in \bar{\Lambda}$, $u \in \mathcal{M}$), then $T(\mathcal{O})$ is also an orbit, and we will say that $\mathcal{O}$ and $T(\mathcal{O})$ are equivalent. The symmetries to be considered below are generated by

- permutations: $T : (M_x, M_y, M_z) \mapsto P(M_x, M_y, M_z)$, $\varphi : (x, y, z) \mapsto P(x, y, z)$ with some $P \in S_3$, where permutations act on (x, y, z) in the standard way, and on the triples (M_x, M_y, M_z) as follows:

$$\begin{aligned} &(123) &&: \big(M_x, M_y, M_z\big) \mapsto \big(M_z, M_x, M_y\big), \\ &(12)(3) &&: \big(M_x, M_y, M_z\big) \mapsto \big(M_y^{-1}, M_x^{-1}, M_z^{-1}\big). \end{aligned}$$

- sign flips: $T : (M_x, M_y, M_z) \mapsto (\varepsilon_x M_x, \varepsilon_y M_y, \varepsilon_z M_z)$, $\varepsilon_{x,y,z} = \pm 1$, $\varphi = id$.

To any orbit O of the induced $\bar{\Lambda}$ action (14) with parameters $\boldsymbol{\omega} \in \mathbb{C}^3$ therefore corresponds a number of equivalent orbits whose parameter triples are obtained from $\boldsymbol{\omega}$ by permutations and the action of the Klein four-group K_4 (by sign changes of two coordinates). By virtue of (10), all these orbits are characterized by the same value of ω_4. To deal with nonequivalent orbits, we quotient the parameter space $\mathbb{C}^3$ by $K_4 \rtimes S_3$, although it is convenient not to fix the fundamental domain explicitly.

Bäcklund transformations (BTs) map solutions of a given Painlevé VI equation to solutions of the same equation with different values of parameters $\theta_{x,y,z,\infty}$. The list of fundamental BTs for PVI is given in the table below, cf. [25]: Here we use the standard notation $\delta = \frac{\theta_x+\theta_y+\theta_z+\theta_\infty}{2}$ and

$$2p = \frac{t(t-1)w'}{w(w-1)(w-t)} - \left(\frac{\theta_x}{w} + \frac{\theta_y}{w-1} + \frac{\theta_z+1}{w-t}\right).$$

Table 1
Bäcklund transformations for Painlevé VI.

	θ_x	θ_y	θ_z	θ_∞	w	t	ω_X	ω_Y	ω_Z	ω_4
s_x	$-\theta_x$	θ_y	θ_z	θ_∞	w	t	ω_X	ω_Y	ω_Z	ω_4
s_y	θ_x	$-\theta_y$	θ_z	θ_∞	w	t	ω_X	ω_Y	ω_Z	ω_4
s_z	θ_x	θ_y	$-\theta_z$	θ_∞	w	t	ω_X	ω_Y	ω_Z	ω_4
s_∞	θ_x	θ_y	θ_z	$2-\theta_\infty$	w	t	ω_X	ω_Y	ω_Z	ω_4
s_δ	$\theta_x-\delta$	$\theta_y-\delta$	$\theta_z-\delta$	$\theta_\infty-\delta$	$w+\frac{\delta}{p}$	t	ω_X	ω_Y	ω_Z	ω_4
r_x	$\theta_\infty-1$	θ_z	θ_y	θ_x+1	t/w	t	ω_X	$-\omega_Y$	$-\omega_Z$	ω_4
r_y	θ_z	$\theta_\infty-1$	θ_x	θ_y+1	$\frac{w-t}{w-1}$	t	$-\omega_X$	ω_Y	$-\omega_Z$	ω_4
r_z	θ_y	θ_x	$\theta_\infty-1$	θ_z+1	$\frac{t(w-1)}{w-t}$	t	$-\omega_X$	$-\omega_Y$	ω_Z	ω_4
P_{xy}	θ_y	θ_x	θ_z	θ_∞	$1-w$	$1-t$	ω_Y	ω_X	ω_Z	ω_4
P_{yz}	θ_x	θ_z	θ_y	θ_∞	w/t	$1/t$	ω_X	ω_Z	ω_Y	ω_4

Remark 9. Five transformations s_ν ($\nu = x, y, z, \infty, \delta$) generate affine Weyl group of type D_4. Using these transformations, one can construct shift operators

$$t_x = s_x s_\delta \left(s_y s_z s_\infty s_\delta\right)^2, \qquad t_y = s_y s_\delta \left(s_x s_z s_\infty s_\delta\right)^2,$$
$$t_z = s_z s_\delta \left(s_x s_y s_\infty s_\delta\right)^2, \qquad t_\infty = s_\infty s_\delta \left(s_x s_y s_z s_\delta\right)^2,$$

acting on the parameter space by simple translations:

	θ_x	θ_y	θ_z	θ_∞
t_x	θ_x+2	θ_y	θ_z	θ_∞
t_y	θ_x	θ_y+2	θ_z	θ_∞
t_z	θ_x	θ_y	θ_z+2	θ_∞
t_∞	θ_x	θ_y	θ_z	$\theta_\infty+2$

Enlarging affine D_4 by the Klein four-group $K_4 \cong \langle r_x, r_y, r_z\rangle$ gives extended affine Weyl group D_4. Full Okamoto affine F_4 action involves additional generators P_{xy}, P_{yz} changing the PVI independent variable t by Möbius transformations of $\mathbb{P}^1$ permuting 0, 1 and ∞.

Last four columns of Table 1 describe the action of BTs on parameters $\omega_{X,Y,Z,4}$ defined by (11)–(13). Observe that all BTs lead to equivalent points in the parameter space of orbits of the induced Λ action (14). We now want to prove a converse statement:

Proposition 10. *Given $\omega_X, \omega_Y, \omega_Z, \omega_4 \in \mathbb{C}$, consider (11)–(13) as a system of equations for unknown $\theta_{x,y,z,\infty}$. Any two solutions of this system are related by the affine D_4 transformations introduced above.*

Proof. Choose an arbitrary solution $\{\theta_\nu^0\}$ ($\nu = x, y, z, \infty$) and denote $p_\nu^0 = 2\cos\pi\theta_\nu^0$. Introduce the auxiliary variable $\xi = p_x^2 + p_y^2 + p_z^2 + p_\infty^2$. It satisfies the cubic equation

$$\xi^3 - a(\omega)\xi^2 + b(\omega)\xi - c(\omega) = 0, \tag{15}$$

where

$$a(\omega) = \omega_4 + 16, \qquad b(\omega) = \omega_X\omega_Y\omega_Z - 4(\omega_X^2 + \omega_Y^2 + \omega_Z^2) + 32\omega_4,$$
$$c(\omega) = \omega_X^2\omega_Y^2 + \omega_X^2\omega_Z^2 + \omega_Y^2\omega_Z^2 - 4\omega_4(\omega_X^2 + \omega_Y^2 + \omega_Z^2) + 16\omega_4^2.$$

Write $\omega_{X,Y,Z,4}$ in terms of $\{p_\nu^0\}$, then three roots of (15) are

$$\xi_0 = \left(p_x^0\right)^2 + \left(p_y^0\right)^2 + \left(p_z^0\right)^2 + \left(p_\infty^0\right)^2,$$
$$\xi_\pm = 8\left(1 + \prod_{\nu=x,y,z,\infty}\cos\pi\theta_\nu^0 \pm \prod_{\nu=x,y,z,\infty}\sin\pi\theta_\nu^0\right).$$

Applying s_δ (or $s_\delta s_x$) to initial solution $\{\theta_\nu^0\}$ gives a solution with $\xi = \xi_-$ (resp. $\xi = \xi_+$). Therefore it is sufficient to prove the proposition for solutions of (11)–(13) with $\xi = \xi_0$.

Assume that at least two of three numbers $\omega_X^2, \omega_Y^2, \omega_Z^2 \in \mathbb{C}$ are distinct, say $\omega_Y^2 \neq \omega_Z^2$. Substituting $\xi = \xi_0$ into easily verified relations

$$(p_x \pm p_\infty)^4 - (\xi \pm 2\omega_X)(p_x \pm p_\infty)^2 + (\omega_Y \pm \omega_Z)^2 = 0$$

we find $(p_x+p_\infty)^2=\left(p_x^0+p_\infty^0\right)^2$ or $\left(p_y^0+p_z^0\right)^2$, $(p_x-p_\infty)^2=\left(p_x^0-p_\infty^0\right)^2$ or $\left(p_y^0-p_z^0\right)^2$. Also if $\xi=\xi_0$ then

$$p_xp_yp_zp_\infty=\omega_4-\xi=p_x^0p_y^0p_z^0p_\infty^0,\qquad p_xp_\infty+p_yp_z=\omega_X=p_x^0p_\infty^0+p_y^0p_z^0,$$

so that $p_xp_\infty=p_x^0p_\infty^0$ or $p_y^0p_z^0$. But now if e.g. $(p_x+p_\infty)^2=\left(p_x^0+p_\infty^0\right)^2$, $(p_x-p_\infty)^2=\left(p_y^0-p_z^0\right)^2$, combining with the latter result we find $\left(p_x^0+p_\infty^0\right)^2=\left(p_y^0+p_z^0\right)^2$ (for $p_xp_\infty=p_y^0p_z^0$) or $\left(p_x^0-p_\infty^0\right)^2=\left(p_y^0-p_z^0\right)^2$ (for $p_xp_\infty=p_x^0p_\infty^0$). Therefore we necessarily have

$$\begin{cases}(p_x+p_\infty)^2=\left(p_x^0+p_\infty^0\right)^2,\\ (p_x-p_\infty)^2=\left(p_x^0-p_\infty^0\right)^2,\end{cases}\quad\text{or}\quad\begin{cases}(p_x+p_\infty)^2=\left(p_y^0+p_z^0\right)^2,\\ (p_x-p_\infty)^2=\left(p_y^0-p_z^0\right)^2.\end{cases}\tag{16}$$

Choose a solution of (16) for p_x and p_∞, then p_y and p_z are unambiguously fixed by

$$(p_\infty\pm p_x)(p_y\pm p_z)=\omega_Y\pm\omega_Z=(p_\infty^0\pm p_x^0)(p_y^0\pm p_z^0)$$

(here we used that $\omega_Y^2\neq\omega_Z^2$). Hence there are 8 possible solutions for (p_x,p_y,p_z,p_∞), namely

$$\begin{aligned}&(\pm p_x^0,\pm p_y^0,\pm p_z^0,\pm p_\infty^0),\quad(\pm p_y^0,\pm p_x^0,\pm p_\infty^0,\pm p_z^0),\\ &(\pm p_z^0,\pm p_\infty^0,\pm p_x^0,\pm p_y^0),\quad(\pm p_\infty^0,\pm p_z^0,\pm p_y^0,\pm p_x^0).\end{aligned}\tag{17}$$

All of them can be obtained from $\left\{p_\nu^0\right\}$ using three affine D_4 transformations $\left(s_xs_ys_zs_\infty s_\delta\right)^2$, $s_\delta s_xs_ys_\delta s_zs_\infty$ and $s_\delta s_xs_zs_\delta s_ys_\infty$. Now given $\{p_\nu\}$, all possible solutions for $\{\theta_\nu\}$ are clearly related by the transformations $\{s_\nu\}$, $\{t_\nu\}$, see Remark 9.

Now let $\omega_X^2=\omega_Y^2=\omega_Z^2$. We can set for definiteness $\omega_X=\omega_Y=\omega_Z$, then three out of four p_ν are equal. Denote this common value by p and let $\tilde{p}$ be the fourth variable. Then

$$\omega_X=p(p+\tilde{p}),\qquad\omega_4=3p^2+\tilde{p}^2+p^3\tilde{p}.\tag{18}$$

Choose a solution $(p^0,\tilde{p}^0)$ of (18). If $\omega_X\neq0$ then the only other solution such that $3p^2+\tilde{p}^2=\xi_0=3\left(p^0\right)^2+\left(\tilde{p}^0\right)^2$ is given by $p=-p^0$, $\tilde{p}=-\tilde{p}^0$. Thus (p_x,p_y,p_z,p_∞) can only be a permutation of $(p^0,p^0,p^0,\tilde{p}^0)$ or $(-p^0,-p^0,-p^0,-\tilde{p}^0)$, which yields at most 8 distinct solutions. As above, all these 4-tuples are related by $\left(s_xs_ys_zs_\infty s_\delta\right)^2$, $s_\delta s_xs_ys_\delta s_zs_\infty$ and $s_\delta s_xs_zs_\delta s_ys_\infty$. Now if $\omega_X=0$ there are 2 possibilities: (1) $p^0=0$, then the only other solution of (18) with the same value of ξ has the form $p=0$, $\tilde{p}=-\tilde{p}^0$; (2) $\tilde{p}^0=-p^0$, then the only such solution is $p=-p^0$, $\tilde{p}=p^0$. Clearly in both cases possible 4-tuples (p_x,p_y,p_z,p_∞) are related by the affine D_4 transformations. $\square$

Remark 11. We have just shown that the map

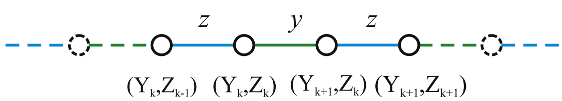
$$\rho:\ \left.\begin{matrix}\text{parameter}\\ \text{space of PVI}\end{matrix}\right/\text{affine } D_4\to\mathbb{C}^4,\qquad[\theta_x,\theta_y,\theta_z,\theta_\infty]\mapsto(\omega_X,\omega_Y,\omega_Z,\omega_4)\tag{19}$$

is injective. Direct calculation shows that ρ is in fact a bijection. Moreover the same result holds true if we replace in (19) affine D_4 by the full affine F_4 action and quotient the set of all triples $(\omega_X,\omega_Y,\omega_Z)$ by $K_4\rtimes S_3$ as described above.

Remark 12. It is more delicate to establish the equivalence of actual PVI solutions as BTs may become singular ($w(t)=0,1,t$ or $p=0$) in the way of transforming a given solution into another one with equivalent parameters.

2.3. 2-colored suborbits

Take a point $\mathbf{r}=(X,Y,Z)\in\mathbb{C}^3$, fix $\boldsymbol{\omega}\in\mathbb{C}^3$ and consider the suborbit $O_{yz}(\mathbf{r})$ of the $\bar{\Lambda}$ action (14), generated from $\mathbf{r}$ by two transformations y and z. Clearly all points of $O_{yz}(\mathbf{r})$ have the same first coordinate X. We set $Y_0=Y$, $Z_0=Z$ and label remaining coordinates as shown on the suborbit graph below.



From (14) one finds a first order linear inhomogeneous difference equation

$$\begin{pmatrix}Y_{k+1}\\ Z_{k+1}\end{pmatrix}=\begin{pmatrix}-1&-X\\ X&X^2-1\end{pmatrix}\begin{pmatrix}Y_k\\ Z_k\end{pmatrix}+\begin{pmatrix}\omega_Y\\ \omega_Z-X\omega_Y\end{pmatrix}.\tag{20}$$

A straightforward computation gives

Lemma 13. *If $X \neq \pm 2$, then the general solution of (20) is*

$$\begin{pmatrix} Y_k \\ Z_k \end{pmatrix} = \frac{1}{\sin\lambda/2}\begin{pmatrix} \sin\frac{(1-2k)\lambda}{2} & -\sin k\lambda \\ \sin k\lambda & \sin\frac{(1+2k)\lambda}{2} \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} + \frac{1}{4-X^2}\begin{pmatrix} 2\omega_Y - X\omega_Z \\ 2\omega_Z - X\omega_Y \end{pmatrix}, \tag{21}$$

where α, β are arbitrary constants and $X = 2\cos\lambda/2$. For $X = \pm 2$ we have

$$\begin{pmatrix} Y_k \\ Z_k \end{pmatrix} = \begin{pmatrix} 1-2k & \mp 2k \\ \pm 2k & 1+2k \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} + \begin{pmatrix} \frac{\omega_Y \pm \omega_Z}{8} - \frac{(\omega_Y \mp \omega_Z)k}{2} + (\omega_Y \mp \omega_Z)k^2 \\ \frac{\omega_Z \pm \omega_Y}{8} + \frac{(\omega_Z \mp \omega_Y)k}{2} + (\omega_Z \mp \omega_Y)k^2 \end{pmatrix}. \tag{22}$$

Now assume that $O_{yz}(\mathbf{r})$ is finite. We call the length of $O_{yz}(\mathbf{r})$ the smallest positive integer N such that $Y_{k+N} = Y_k$, $Z_{k+N} = Z_k$. Since x, y, z are involutions, the graph of any 2-colored finite suborbit can only be a simple cycle (as the length 2 yz-suborbit 2-3-4-5 in Example 8) or a line with a self-loop at each of its ends (as e.g. the length 3 xz-suborbit 1-2-3 or the length 2 xy-suborbit 3-4 of the same example).

Lemma 14. *Let N be the length of $O_{yz}(\mathbf{r})$. If $N > 1$, then $X = 2\cos\pi n_X/N$, where n_X is an integer relatively prime to N satisfying $0 < n_X < N$.*

Proof. Let $X \neq \pm 2$ and impose $Y_{k+N} = Y_k$, $Z_{k+N} = Z_k$ in (21). This gives $\sin\frac{N\lambda}{2} = 0$, otherwise $\alpha = \beta = 0$ and hence $N = 1$. Therefore $\lambda = 2\pi n_X/N$, $n_X \in \mathbb{Z}$ and we can choose $0 < n_X < N$. Clearly n_X and N are coprime; otherwise N is not the smallest period of (21).

Now if $X = \pm 2$, then substituting $Y_{k+N} = Y_k$, $Z_{k+N} = Z_k$ into (22) we find two conditions: (1) $\omega_Y \mp \omega_Z = 0$ and (2) $\alpha \pm \beta = 0$. This in turn implies that $Y_k = \text{const}$, $Z_k = \text{const}$, i.e. $O_{yz}(\mathbf{r})$ consists of a single point. □

Definition 15. Let $O \subset \mathbb{C}^3$ be an orbit of the $\bar{\Lambda}$ action (14). A point $\mathbf{r} \in O$ is called *good* if it is not fixed by at least two of three transformations x, y, z; otherwise we say that $\mathbf{r}$ is a *bad* point.

The case when the whole orbit consists of a single point is trivial. Hence below by a bad point we most often mean a point fixed by two transformations. The orbit graph has then two self-loops at the corresponding vertex.

Example 16. The point 1 in Example 8 is bad, and the others are good.

Lemma 17. *Let $O \subset \mathbb{C}^3$ be a finite orbit of (14). If $\mathbf{r} = (X, Y, Z) \in O$ is a good point, then*

$$X = 2\cos\pi r_X, \qquad Y = 2\cos\pi r_Y, \qquad Z = 2\cos\pi r_Z, \tag{23}$$

where $r_{X,Y,Z} \in \mathbb{Q}$ and $0 < r_{X,Y,Z} < 1$. If $\mathbf{r} \in O$ is a bad point, fixed by y and z but not by x, then (23) still holds for Y and Z.

Proof. If $\mathbf{r}$ is not fixed by x, then the lengths of xz- and xy-suborbits of $\mathbf{r}$ are strictly greater than 1. If $\mathbf{r}$ is good the same is true for each of the three 2-colored suborbits of $\mathbf{r}$. Both statements then follow from Lemma 14. □

2.4. Main technical lemma

This subsection is devoted to a technical result to be extensively used later. Namely, we want to find all rational solutions of the equation

$$\sum_{j=1}^{n} \cos 2\pi\varphi_j = 0 \tag{24}$$

with $n \leq 6$. Without loss of generality we assume that $0 \leq \varphi_j < 1$ and consider the n-tuples $(\varphi_1, \ldots, \varphi_n)$ related by permutations, transformations $\varphi_j \to 1 - \varphi_j$ and by the simultaneous change $\varphi_j \to 1/2 - \varphi_j$ as equivalent.

Definition 18. A rational n-tuple $(\varphi_1, \ldots, \varphi_n)$ is called irreducible if it satisfies (24) and $\sum_{j\in E}\cos 2\pi\varphi_j \neq 0$ for any proper subset $E \subset \{1, \ldots, n\}$.

It then suffices to classify irreducible n-tuples $(\varphi_1, \ldots, \varphi_n)$ with $n \leq 6$. We first prove an auxiliary result concerning rational solutions of the equation

$$\sum_{j=1}^{n} e^{2\pi i\varphi_j} = 0. \tag{25}$$

Again we can assume that $0 \leq \varphi_j < 1$ and consider the solution n-tuples up to permutations. Also note that the shift of all φ_j by a common phase $\varphi \in \mathbb{Q}$ yields another solution.

Lemma 19. *All inequivalent irreducible (in the sense that $\sum_{j\in E} e^{2\pi i\varphi_j} \neq 0$ for any proper subset $E \subset \{1,\dots,n\}$) rational n-tuples with $n \leq 6$ solving (25) are given by*

- *the* 6*-tuple*

$$\left(\varphi-\frac{1}{6},\varphi+\frac{1}{6},\varphi+\frac{1}{5},\varphi+\frac{2}{5},\varphi+\frac{3}{5},\varphi+\frac{4}{5}\right), \tag{26}$$

- *the* 5*-tuple*

$$\left(\varphi,\varphi+\frac{1}{5},\varphi+\frac{2}{5},\varphi+\frac{3}{5},\varphi+\frac{4}{5}\right), \tag{27}$$

- *the triple $\left(\varphi,\varphi+\frac{1}{3},\varphi+\frac{2}{3}\right)$ and the pair $\left(\varphi,\varphi+\frac{1}{2}\right)$, with $\varphi\in\mathbb{Q}$.*

Proof. First part of the proof follows [26,5]. Write $\varphi_k = \frac{n_k}{d_k}$, where $k = 1,\dots,n$ $(1 < n \leq 6)$ and d_k, n_k are either positive coprime integers with $d_k > n_k$ or $n_k = 0$. Let p be a prime which is a divisor of at least one of $d_1,\dots,d_n$, and denote by δ_k, l_k, c_k, ν_k the integers such that

$$d_k = \delta_k p^{l_k}, \qquad n_k = c_k\delta_k + \nu_k p^{l_k},$$

where δ_k is prime to p, $0 \leq c_k < p^{l_k}$; c_k is prime to p for $l_k \neq 0$, otherwise $c_k = 0$. Then

$$\varphi_k = f_k + \frac{c_k}{p^{l_k}}, \qquad f_k = \frac{\nu_k}{\delta_k}.$$

Reordering $\varphi_1,\dots,\varphi_n$ so that $l_1 \geq l_2 \geq \cdots \geq l_n$, we define the function

$$g_k(x) = \begin{cases} e^{2\pi i f_k} x^{c_k p^{l_1-l_k}} & \text{if } c_k \neq 0,\\ e^{2\pi i\varphi_k} & \text{if } c_k = 0,\end{cases}$$

and the polynomial

$$U(x) = \sum_{k=1}^{n} g_k(x). \tag{28}$$

By construction $g_k\left(\exp\left(\frac{2\pi i}{p^{l_1}}\right)\right) = e^{2\pi i\varphi_k}$, and (25) then implies that $U\left(\exp\left(\frac{2\pi i}{p^{l_1}}\right)\right) = 0$.

It is known since 1854 [27] that the polynomial

$$P(x) = 1 + x^{p^{l_1-1}} + x^{2p^{l_1-1}} + \cdots + x^{(p-1)p^{l_1-1}}$$

is irreducible in the ring of polynomials with coefficients in any extension of the form $\mathbb{Q}(\zeta_1,\dots,\zeta_m)$, where ζ_j is a root of unity of the order coprime with p. Since $P\left(\exp\left(\frac{2\pi i}{p^{l_1}}\right)\right) = 0$, then either (a) $U(x) \equiv 0$ or (b) $U(x) \not\equiv 0$ is divisible by $P(x)$.

Case (a). The powers $c_k p^{l_1-l_k}$, appearing in the functions $g_k(x)$, are all equal. Otherwise one could write $U(x)$ as a sum of at least two polynomials equal to 0, and the irreducibility condition fails. Therefore $l_k = l_1$, $c_k = c_1$. Now it is sufficient to subtract the common phase $\frac{c_1}{p^{l_1}}$ from all φ_k to eliminate p from all denominators.

Case (b). Write $U(x) = P(x)Q(x)$. The degree of $U(x)$ is at most $p^{l_1}-1$, hence the degree of $Q(x)$ is at most $p^{l_1-1}-1$. Then the numbers N_U and N_Q of different powers of x in $U(x)$ and $Q(x)$ must be related by $N_U = pN_Q$. In particular, since in our case $N_U \leq 6$, the prime p can only be equal to 2, 3 or 5.

The powers $c_k p^{l_1-l_k}$ are all equal modulo p^{l_1-1} to s, where s is some integer independent of k, $0 \leq s < p^{l_1-1}$. Otherwise one could collect powers corresponding to different s and write $U(x)$ as a sum of at least two polynomials, each of them either divisible by $P(x)$ or vanishing identically. Corresponding n-tuple is then reducible, therefore we can only have $N_Q = 1$, $Q(x) = \alpha x^s$.

Suppose that $l_1 \geq 2$. Since c_1 is prime to p, s is also prime to p and all n powers of x that appear in the functions $g_k(x)$ are not divisible by p^{l_1-1} and by p; in particular, all c_k are non-zero. This in turn implies that $l_k = l_1$ for any k. Now $c_k = s + N_k p^{l_1-1}$ and subtracting from all φ_k the common phase $\frac{s}{p^{l_1}}$ eliminates all higher (greater than 1) powers of p from the denominators.

It remains to consider $l_1 = 1$, $p = 2, 3$ or 5:

(b.1) Let $l_1 = 1$, $p = 5$, then $n = 5$ or 6. If $n = 6$, then from $U(x) = \alpha x^s P(x)$ four out of six phases are equal, say $f_1 = f_2 = f_3 = f_4$, and the remaining two satisfy $e^{2\pi i f_5} + e^{2\pi i f_6} = e^{2\pi i f_1}$. Setting $f_1 = 0$ gives $f_5 = \frac{1}{6}$, $f_6 = -\frac{1}{6}$, then $(c_1, c_2, c_3, c_4, c_5 = c_6)$ is a permutation of $(0, 1, 2, 3, 4)$ and we obtain the 6-tuple (26).

If $n = 5$, then $f_1 = f_2 = f_3 = f_4 = f_5$, $(c_1, c_2, c_3, c_4, c_5)$ is a permutation of $(0, 1, 2, 3, 4)$, which leads to the 5-tuple (27).

(b.2) Now every φ_k can only be equal to 0, $\frac{1}{2}$, $\pm\frac{1}{3}$ or $\pm\frac{1}{6}$. Direct check shows that the only irreducible n-tuples with $n \leq 6$ that can be built from such numbers are (equivalent to) the triple $\left(0, \frac{1}{3}, -\frac{1}{3}\right)$ and the pair $\left(0, \frac{1}{2}\right)$. □

We now establish a similar classification of rational solutions of (24):

Lemma 20. *Inequivalent irreducible rational n-tuples solving (24) with $1 < n \leq 6$ fall into one of the following classes:*

- 13 *nontrivial irreducible 6-tuples*

$$\left(\frac{1}{11},\frac{2}{11},\frac{3}{11},\frac{4}{11},\frac{5}{11},\frac{1}{6}\right), \tag{VI$_1$}$$

$$\left(\frac{L}{7}+\frac{1}{6},\frac{L}{7}-\frac{1}{6},\frac{2L}{7},\frac{3L}{7},0,\frac{1}{3}\right),\quad L=1,2,3, \tag{VI$_2$}$$

$$\left(\frac{L}{7}+\frac{1}{6},\frac{L}{7}-\frac{1}{6},\frac{2L}{7},\frac{3L}{7},\frac{1}{10},\frac{3}{10}\right),\quad L=1,2,3, \tag{VI$_3$}$$

$$\left(\frac{L}{7}+\frac{1}{6},\frac{L}{7}-\frac{1}{6},\frac{2L}{7}+\frac{1}{6},\frac{2L}{7}-\frac{1}{6},\frac{3L}{7},\frac{1}{6}\right),\quad L=1,2,3, \tag{VI$_4$}$$

$$\left(\frac{1}{7},\frac{2}{7},\frac{3}{7},0,\frac{1}{5},\frac{2}{5}\right),\left(\frac{1}{7},\frac{2}{7},\frac{3}{7},\frac{1}{15},\frac{4}{15},\frac{3}{10}\right),\left(\frac{1}{7},\frac{2}{7},\frac{3}{7},\frac{1}{10},\frac{2}{15},\frac{7}{15}\right), \tag{VI$_5$}$$

and an infinite family of the form

$$\left(\varphi+\frac{1}{6},\varphi-\frac{1}{6},\varphi+\frac{1}{5},\varphi+\frac{2}{5},\varphi+\frac{3}{5},\varphi+\frac{4}{5}\right),\quad \varphi\in\mathbb{Q}, \tag{VI$_\varphi$}$$

- 7 *nontrivial irreducible* 5*-tuples*

$$\left(0,\frac{1}{30},\frac{1}{3},\frac{11}{30},\frac{2}{5}\right),\qquad \left(0,\frac{1}{5},\frac{7}{30},\frac{1}{3},\frac{13}{30}\right), \tag{V$_1$}$$

$$\left(\frac{L}{7}+\frac{1}{6},\frac{L}{7}-\frac{1}{6},\frac{2L}{7},\frac{3L}{7},\frac{1}{6}\right),\quad L=1,2,3, \tag{V$_2$}$$

$$\left(\frac{1}{7},\frac{2}{7},\frac{3}{7},0,\frac{1}{3}\right),\qquad \left(\frac{1}{7},\frac{2}{7},\frac{3}{7},\frac{1}{10},\frac{3}{10}\right), \tag{V$_3$}$$

and an infinite family of the form

$$\left(\varphi,\varphi+\frac{1}{5},\varphi+\frac{2}{5},\varphi+\frac{3}{5},\varphi+\frac{4}{5}\right),\quad \varphi\in\mathbb{Q}, \tag{V$_\varphi$}$$

- 4 *nontrivial irreducible quadruples*

$$\left(0,\frac{1}{5},\frac{1}{3},\frac{2}{5}\right),\ \left(\frac{1}{30},\frac{1}{6},\frac{11}{30},\frac{2}{5}\right),\ \left(\frac{1}{15},\frac{4}{15},\frac{3}{10},\frac{1}{3}\right),\ \left(\frac{1}{7},\frac{2}{7},\frac{3}{7},\frac{1}{6}\right), \tag{IV}$$

- 1 *nontrivial irreducible triple*

$$\left(\frac{1}{10},\frac{3}{10},\frac{1}{3}\right) \tag{III$_1$}$$

and an infinite family of the form

$$\left(\varphi,\varphi+\frac{1}{3},\varphi-\frac{1}{3}\right),\quad \varphi\in\mathbb{Q}, \tag{III$_\varphi$}$$

- *an infinite family of pairs of the form*

$$\left(\varphi,\frac{1}{2}-\varphi\right),\quad \varphi\in\mathbb{Q}. \tag{II$_\varphi$}$$

Proof. We use the same ideas, notations and conventions as in the proof of Lemma 19. One modification concerns the functions $g_k(x)$ which are now defined by

$$g_k(x)=\begin{cases}\dfrac{1}{2}\left[e^{2\pi i f_k}x^{c_k p^{l_1-l_k}}+e^{-2\pi i f_k}x^{p^{l_1}-c_k p^{l_1-l_k}}\right] & \text{if } c_k\neq 0,\\ \cos 2\pi\varphi_k & \text{if } c_k=0.\end{cases}$$

As $g_k\left(\exp\left(\frac{2\pi i}{p^{l_1}}\right)\right) = \cos 2\pi\varphi_k$, one has again $U\left(\exp\left(\frac{2\pi i}{p^{l_1}}\right)\right) = 0$, so that either (a) $U(x) \equiv 0$ or (b) $U(x) \not\equiv 0$ is divisible by $P(x)$.

Case (a). All $2n$ powers $c_k p^{l_1-l_k}$, $p^{l_1} - c_k p^{l_1-l_k}$, appearing in the functions $g_k(x)$, are simultaneously divisible or non-divisible by p unless we have a reducible n-tuple. Since c_1 is prime to p, they are actually non-divisible, which in turn gives $l_k = l_1$ for any k. Irreducibility then implies that c_k can only be equal to c_1 or $p^{l_1} - c_1$. In fact we can assume that $c_k = c_1$, as the transformation $\varphi_k \mapsto 1 - \varphi_k$ maps $f_k \mapsto -f_k$, $c_k \mapsto p^{l_k} - c_k$. Now one has

$$U(x) = \frac{1}{2}x^{c_1}\sum_{k=1}^{n} e^{2\pi i f_k} + \frac{1}{2}x^{p^{l_1}-c_1}\sum_{k=1}^{n} e^{-2\pi i f_k} = 0,$$

and, since $c_1 \neq p^{l_1} - c_1$ except in the trivial case $p = 2$, $l_1 = 1$, the problem is reduced to the classification of rational solutions of Eq. (25), given by Lemma 19.

Case (b). Set $U(x) = P(x)Q(x)$, then by the same reasoning as above $N_U = pN_Q$. However, here $N_U \leq 12$, therefore p can be equal to 2, 3, 5, 7 or 11.

$2n$ powers $c_k p^{l_1-l_k}$, $p^{l_1} - c_k p^{l_1-l_k}$ are all equal modulo p^{l_1-1} to s or $p^{l_1-1} - s$, where the integer s does not depend on k, $0 \leq s < p^{l_1-1}$. Otherwise one could collect powers corresponding to different s and write $U(x)$ as a sum of at least two polynomials, each of them either divisible by $P(x)$ or vanishing. Since $p^{l_1} - c_k p^{l_1-l_k} = -c_k p^{l_1-l_k} \mod p^{l_1-1}$, both terms coming from a given $g_k(x)$ will appear in the same polynomial, and then the corresponding n-tuple is reducible. Hence N_Q can only be equal to 1 or 2.

If $l_1 \geq 2$, then all $2n$ powers of x that appear in the functions $g_k(x)$ are not divisible by p and therefore $l_k = l_1$ for any k.

Two powers c_k and $p^{l_1} - c_k$ are distinct modulo p^{l_1-1} for all but a finite number of values of l_1 and p. Indeed, if they are the same, one has $2c_k = 0 \mod p^{l_1-1}$. However, this is impossible for $p \geq 3$, $l_1 \geq 2$ and for $p = 2$, $l_1 \geq 3$, since all c_k are prime to p. Let us now consider separately two cases:

(b.1) $p \geq 3$, $l_1 \geq 2$ or $p = 2$, $l_1 \geq 3$;
(b.2) $p = 3, 5, 7, 11$, $l_1 = 1$ or $p = 2$, $l_1 = 1, 2$.

(b.1) When $c_k \neq p^{l_1} - c_k \mod p^{l_1-1}$, we use $N_Q \leq 2$ to write the relation $U(x) = P(x)Q(x)$ as two distinct equations containing different (mod p^{l_1-1}) powers of x. Replacing $\varphi_k \mapsto 1-\varphi_k$ if necessary, one finds that both equations are equivalent to the following one:

$$\sum_{j=1}^{n} e^{2\pi i f_k} x^{c_k} = \alpha x^s P(x), \quad \alpha \neq 0. \tag{29}$$

Assume that $n = 6$. It is impossible to satisfy (29) if $p = 7, 11$. For $p = 5$ four out of six phases are equal, say $f_1 = f_2 = f_3 = f_4$, and the remaining two satisfy

(b.1.1) $e^{2\pi i f_5} + e^{2\pi i f_6} = e^{2\pi i f_1}$.

In addition we have $c_k = s + N_k \cdot 5^{l_1-1}$, where $(N_1, N_2, N_3, N_4, N_5 = N_6)$ is a permutation of $(0, 1, 2, 3, 4)$. Now applying Lemma 19 to find rational solutions of (b.1.1) we see that resulting 6-tuples are of type (VI_φ).

For $p = 3$, up to permutations there are only three possibilities:

(b.1.2) $e^{2\pi i f_1} = e^{2\pi i f_2} = e^{2\pi i f_3} + e^{2\pi i f_4} + e^{2\pi i f_5} + e^{2\pi i f_6}$,
(b.1.3) $e^{2\pi i f_1} = e^{2\pi i f_2} + e^{2\pi i f_3} = e^{2\pi i f_4} + e^{2\pi i f_5} + e^{2\pi i f_6}$,
(b.1.4) $e^{2\pi i f_1} + e^{2\pi i f_2} = e^{2\pi i f_3} + e^{2\pi i f_4} = e^{2\pi i f_5} + e^{2\pi i f_6} \neq 0$.

Finally, for $p = 2$ one should have one of the following:

(b.1.5) $e^{2\pi i f_1} = e^{2\pi i f_2} + e^{2\pi i f_3} + e^{2\pi i f_4} + e^{2\pi i f_5} + e^{2\pi i f_6}$,
(b.1.6) $e^{2\pi i f_1} + e^{2\pi i f_2} = e^{2\pi i f_3} + e^{2\pi i f_4} + e^{2\pi i f_5} + e^{2\pi i f_6} \neq 0$,
(b.1.7) $e^{2\pi i f_1} + e^{2\pi i f_2} + e^{2\pi i f_3} = e^{2\pi i f_4} + e^{2\pi i f_5} + e^{2\pi i f_6} \neq 0$.

In each of these cases the problem is reduced to Lemma 19. The 6-tuples we obtain at the end turn out to be reducible or belong to the family (VI_φ).

Other possibilities ($n = 3, 4, 5$) can be treated in a similar manner. They lead to 5-tuples of type (V_φ) and triples of type (III_φ).

(b.2) We first consider the case when the denominator of every φ_k ($k = 1, \ldots, n$) is not divisible by 7 and 11:

Lemma 21. *Inequivalent irreducible n-tuples solving (24) with $3 \leq n \leq 6$ such that every d_k ($k = 1, \ldots, n$) is a divisor of $2^2 \cdot 3 \cdot 5 = 60$ are given by*

- 6-*tuples:*

$$\left(0, \frac{1}{30}, \frac{1}{5}, \frac{11}{30}, \frac{2}{5}, \frac{2}{5}\right), \quad \left(0, \frac{1}{30}, \frac{7}{30}, \frac{1}{3}, \frac{11}{30}, \frac{13}{30}\right), \quad \left(0, \frac{1}{5}, \frac{1}{5}, \frac{7}{30}, \frac{2}{5}, \frac{13}{30}\right),$$
$$\left(\frac{1}{60}, \frac{1}{60}, \frac{13}{60}, \frac{7}{20}, \frac{23}{60}, \frac{5}{12}\right), \quad \left(\frac{1}{60}, \frac{1}{20}, \frac{11}{60}, \frac{23}{60}, \frac{23}{60}, \frac{5}{12}\right), \quad \left(\frac{1}{60}, \frac{11}{60}, \frac{13}{60}, \frac{13}{60}, \frac{5}{12}, \frac{9}{20}\right),$$
$$\left(\frac{1}{12}, \frac{7}{60}, \frac{17}{60}, \frac{19}{60}, \frac{19}{60}, \frac{7}{20}\right).$$

- 5-*tuples:*

$$\left(0, \frac{1}{5}, \frac{1}{5}, \frac{2}{5}, \frac{2}{5}\right), \quad \left(\frac{1}{60}, \frac{11}{60}, \frac{13}{60}, \frac{23}{60}, \frac{5}{12}\right), \quad \left(\frac{1}{30}, \frac{1}{6}, \frac{7}{30}, \frac{11}{30}, \frac{13}{30}\right),$$
$$\left(0, \frac{1}{30}, \frac{1}{3}, \frac{11}{30}, \frac{2}{5}\right), \quad \left(0, \frac{1}{5}, \frac{7}{30}, \frac{1}{3}, \frac{13}{30}\right).$$

- *quadruples:*

$$\left(0, \frac{1}{5}, \frac{1}{3}, \frac{2}{5}\right), \quad \left(\frac{1}{30}, \frac{1}{6}, \frac{11}{30}, \frac{2}{5}\right), \quad \left(\frac{1}{15}, \frac{4}{15}, \frac{3}{10}, \frac{1}{3}\right).$$

- *triples:*

$$\left(0, \frac{1}{3}, \frac{1}{3}\right), \quad \left(\frac{1}{60}, \frac{19}{60}, \frac{7}{20}\right), \quad \left(\frac{1}{30}, \frac{3}{10}, \frac{11}{30}\right), \quad \left(\frac{1}{20}, \frac{17}{60}, \frac{23}{60}\right),$$
$$\left(\frac{1}{15}, \frac{4}{15}, \frac{2}{5}\right), \quad \left(\frac{1}{10}, \frac{3}{10}, \frac{1}{3}\right).$$

Proof. Direct (e.g., Mathematica) computation. Notice that all obtained 6-tuples, first three 5-tuples and all but the last triple belong to the infinite families (VI_φ), (V_φ) and (III_φ), respectively. $\square$

The case $p = 11$, $l_1 = 1$ is possible only for $n = 6$. We have $N_Q = 1$, $\deg Q = 0$, hence $Q(x) = \alpha$, $N_U = 11$, $\deg U = 10$ and, consequently, one can choose $l_1 = \cdots = l_5 = 1$, $l_6 = 0$, $c_k = k$ $(k = 1, \ldots, 5)$, $c_6 = 0$. This gives the irreducible 6-tuple (VI_1).

Remaining case $p = 7$, $l_1 = 1$ is possible only for $n = 4, 5, 6$. Similarly to the above, $N_Q = 1$, $\deg Q = 0$, $Q(x) = \alpha$, $N_U = 7$, $\deg U = 6$, and in addition for all $k = 2, \ldots, n$ either $l_k = 1$ or $c_k = 0$. For $n = 6$ one then has four possibilities:

- $(c_1, c_2, c_3 = c_4)$ is a permutation of $(1, 2, 3)$, $c_5 = c_6 = 0$; this gives $f_1 = f_2 = 0$ and
$$e^{2\pi i f_3} + e^{2\pi i f_4} = 2\cos 2\pi f_5 + 2\cos 2\pi f_6 = 1. \tag{30}$$
Recall that $f_1, \ldots, f_6$ are rational numbers with denominator which is a divisor of 60. Using Lemma 21 to classify the appropriate solutions of (30), one finds that the only irreducible 6-tuples obtained in this way are given by (VI_2) and (VI_3).
- $(c_1, c_2 = c_3, c_4 = c_5)$ is a permutation of $(1, 2, 3)$, $c_6 = 0$; then
$$f_1 = 0, \quad e^{2\pi i f_2} + e^{2\pi i f_3} = e^{2\pi i f_4} + e^{2\pi i f_5} = 2\cos 2\pi f_6 = 1,$$
which leads to the family of irreducible 6-tuples (VI_4).
- $(c_1, c_2, c_3 = c_4 = c_5)$ is a permutation of $(1, 2, 3)$, $c_6 = 0$; then $f_1 = f_2 = 0$ and
$$e^{2\pi i f_3} + e^{2\pi i f_4} + e^{2\pi i f_5} = 2\cos 2\pi f_6 = 1.$$
All 6-tuples arising here turn out to be reducible.
- $(c_1, c_2, c_3) = (1, 2, 3)$, $c_4 = c_5 = c_6 = 0$, which implies $f_1 = f_2 = f_3 = 0$ and
$$2\cos 2\pi f_4 + 2\cos 2\pi f_5 + 2\cos 2\pi f_6 = 1. \tag{31}$$
Using again Lemma 21 to find irreducible solutions of (31), we obtain 3 irreducible 6-tuples (VI_5).

For $n = 5$, there are two possibilities:

- $(c_1, c_2, c_3 = c_4)$ is a permutation of $(1, 2, 3)$, $c_5 = 0$; this implies $f_1 = f_2 = 0$ and
$$e^{2\pi i f_3} + e^{2\pi i f_4} = 2\cos 2\pi f_5 = 1,$$
so that we find 3 irreducible 5-tuples (VI_2).

- $(c_1, c_2, c_3) = (1, 2, 3)$, $c_4 = c_5 = 0$, hence $f_1 = f_2 = f_3 = 0$ and
$$2\cos 2\pi f_4 + 2\cos 2\pi f_5 = 1.$$
This gives 2 irreducible 5-tuples (V_3).

Finally, for $n = 4$ we should have $(c_1, c_2, c_3) = (1, 2, 3)$, $c_4 = 0$ and, therefore, $f_1 = f_2 = f_3 = 0$, $2\cos 2\pi f_4 = 1$, which leads to the fourth irreducible quadruple in (IV). This concludes the proof of Lemma 20. □

Remark 22. The classification of irreducible rational solutions of (24) with $n \le 4$ is essentially equivalent to Lemma 1.13 in [5]. In fact we will see shortly that this partial result is already sufficient to find all finite $\bar{\Lambda}$ orbits with $\omega_X^2 \ne \omega_Y^2 \ne \omega_Z^2$. Its extension to $n = 5, 6$ is needed to treat the case when $\boldsymbol{\omega} \in \mathbb{C}^3$ is fixed by some of the $K_4 \rtimes S_3$ transformations.

2.5. Bounds on suborbit lengths

Let $O \subset \mathbb{C}^3$ be a finite orbit of the induced $\bar{\Lambda}$ action (14). We choose an arbitrary 2-colored suborbit $O_{yz} \subset O$ (i.e. the suborbit generated from a given point by two transformations y and z), denote its length by N and label the points of O_{yz} as in Section 2.3.

Throughout this subsection we assume that $N > 1$. Denote $X = 2\cos\lambda/2$, then by Lemma 14 one has $\lambda = 2\pi r_X$, $r_X = n_X/N$, where $n_X \in \mathbb{Z}$ is prime to N and we choose $0 < n_X < N$. Lemma 13 implies in addition that Y_k, Z_k $(k = 0, 1, \ldots, N-1)$ are given by (21).

When the graph of O_{yz} is a simple cycle, it contains $2N$ points and all of them are good. Then by Lemma 17 for $k = 0, \ldots, N-1$ we have

$$Y_k = 2\cos\pi r_{Y_k}, \qquad Z_k = 2\cos\pi r_{Z_k}, \quad r_{Y_k}, r_{Z_k} \in \mathbb{Q}, 0 < r_{Y_k}, r_{Z_k} < 1. \tag{32}$$

If $\Sigma(O_{yz})$ is a line with self-loops at the ends, then there are N distinct points. While two endpoints can in principle be bad, the other $N-2$ points are good so that their coordinates satisfy (32).

Lemma 23. *Two distinct vertices of $\Sigma(O_{yz})$ characterized by the same coordinate Y (or Z) are necessarily connected by an edge of color z (resp. y).*

Proof. Let (X, Y, Z) be an arbitrary point in O. Since $\omega_{X,Y,Z,4}$ are fixed by the $\bar{\Lambda}$ action, the quantity

$$XYZ + X^2 + Y^2 + Z^2 - \omega_X X - \omega_Y Y - \omega_Z Z = \text{const} = 4 - \omega_4$$

is an orbit invariant. Computing this invariant for two distinct points (X, Y, Z), (X, Y, Z') in O_{yz} we find $Z' = \omega_Z - Z - XY = z(Z)$. □

Remark 24. In the simple cycle case, Lemma 23 implies that $Y_k \ne Y_{k'}$, $Z_k \ne Z_{k'}$ for $k \ne k'$ where $k, k' = 0, \ldots, N-1$. Similarly, in the line case for any k there exists at most one $k' \ne k$ such that $Y_k = Y_{k'}$ (or $Z_k = Z_{k'}$).

Lemma 25. *The coordinates $\{Y_k\}$, $\{Z_k\}$ satisfy the following identities:*

$$\text{for } N \text{ even, } n_X \text{ odd: } \begin{cases} Y_k + Y_{k+N/2} = p_+ + p_-, \\ Z_k + Z_{k+N/2} = p_+ - p_-, \end{cases} \tag{33}$$

$$\text{for } N \text{ odd, } n_X \text{ even: } \quad Y_k + Z_{k+(N-1)/2} = p_+, \tag{34}$$

$$\text{for } N \text{ odd, } n_X \text{ odd: } \quad Y_k - Z_{k+(N-1)/2} = p_-, \tag{35}$$

where $k = 0, \ldots, N-1$ and $p_\pm = \frac{\omega_Y \pm \omega_Z}{2 \pm X}$.

Proof. Straightforward substitution of (21) into (33)–(35). □

Proposition 26. *If N is even and at least one of two parameters ω_Y, ω_Z is different from 0, then $N \le 10$.*

Proof. When at least one of ω_Y, ω_Z differs from 0, at least one of $p_+ \pm p_-$ is also non-zero. Assume for definiteness that $p_+ + p_- \ne 0$ and consider the first equation in (33). It implies that for any $k, k' = 0, \ldots, N-1$ one has

$$Y_k + Y_{k+N/2} = Y_{k'} + Y_{k'+N/2} \ne 0. \tag{36}$$

First assume that the graph of O_{yz} is a simple cycle. All Y_k are then distinct and have the form (32). Hence (36) reduces to an equation of type (24) with $n = 4$, whose rational solutions have been classified in Lemma 20. We now consider different types of solutions to maximize the number N^c of possible unordered couples $\left(Y_k, Y_{k+N/2}\right)$ of the form (32), characterized by the same value of $Y_k + Y_{k+N/2}$:

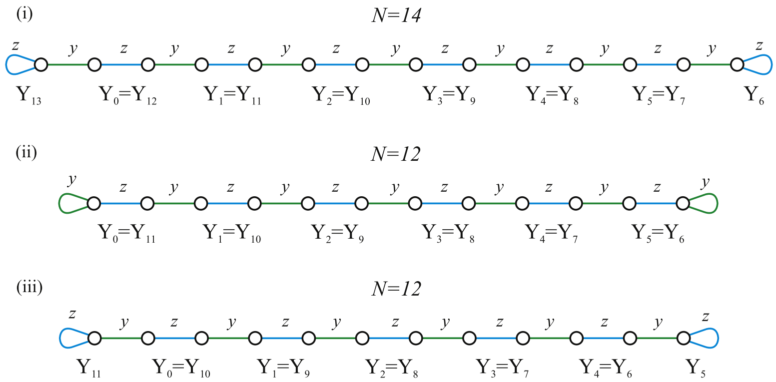


Fig. 2. Three possible graphs for $N = 12, 14$.

- Splitting of the rational solution quadruple into two (not necessarily irreducible) pairs is possible only for $k' = k$ or $k' = k + N/2$, therefore one should not take such solutions into account when computing N^c (here we used that $p_+ + p_- \neq 0$!).
- Assume that $Y_{k_0} = 0$ for some k_0, then for any k one has $Y_k + Y_{k+N/2} = Y_{k_0+N/2}$. This is an equation of type (24) with $n = 3$. By Lemma 20, if $Y_{k_0+N/2} \neq \pm 1, \pm 2\cos\pi/5, \pm 2\cos 2\pi/5$, the only possible couple different from $\left(0, Y_{k_0+N/2}\right)$ is
$$\left(2\cos\pi(r_{Y_{k_0+N/2}} + 1/3), 2\cos\pi(r_{Y_{k_0+N/2}} - 1/3)\right)$$
and therefore $N^c = 2$. When $Y_{k_0+N/2} = \pm 1$, the only compatible couple is $\left(\pm 2\cos\pi/5, \mp 2\cos 2\pi/5\right)$ so that again $N^c = 2$.
Finally, for (a) $Y_{k_0+N/2} = \pm 2\cos\pi/5$ and (b) $Y_{k_0+N/2} = \pm 2\cos 2\pi/5$ one has $N^c = 3$ as in both cases we have three compatible couples:
(a) $\left(0, \pm 2\cos\pi/5\right)$, $\left(\pm 1, \pm 2\cos 2\pi/5\right)$, $\left(\pm 2\cos 2\pi/15, \pm 2\cos 8\pi/15\right)$;
(b) $\left(0, \pm 2\cos 2\pi/5\right)$, $\left(\mp 1, \pm 2\cos\pi/5\right)$, $\left(\pm 2\cos\pi/15, \pm 2\cos 11\pi/15\right)$.
- If there is no Y_k equal to zero, the solution quadruple can only be equivalent to one of the last 3 quadruples in (IV) (first quadruple is excluded because $Y_k \neq \pm 2$). Direct check then shows that for any choice of $\left(Y_k, Y_{k+N/2}\right)$ there is only one compatible couple, i.e. $N^c = 2$.

Since the maximal possible value of N^c is 3, even length N of the simple cycle cannot exceed 6.

When the graph of $\mathcal{O}_{yz}$ is a line, the same reasoning shows that $N \leq 14$, otherwise the number of distinct compatible couples $\left(Y_k, Y_{k+N/2}\right)$ satisfying (32) is greater than 3. We now want to improve this bound to $N \leq 10$ using that for $N = 12, 14$ the number of such couples is 3 and therefore Y-coordinates of good points should give (a) or (b) above.

In Fig. 2 we show three possible graphs and label each vertex by its Y-coordinate. Third diagram (iii) can in fact be immediately excluded, since in this case $2Y_2 = Y_1 + Y_3 = Y_0 + Y_4$ but no couple in (a) or (b) contains two equal cosines. To exclude the remaining two cases, use that from (20) follows a 2nd order difference equation for $\{Y_k\}$:
$$Y_{k+2} + (2 - X^2)Y_{k+1} + Y_k = 2\omega_Y - X\omega_Z.$$
It implies in particular that for both (i) and (ii) we should have
$$X^2 - 1 = \frac{Y_4 - Y_1}{Y_3 - Y_2}. \tag{37}$$
Since (Y_1, Y_4) and (Y_2, Y_3) are necessarily given by two couples from (a) or (b), the RHS of (37) can only take one of 12 values
$$\varepsilon_1(\sqrt{5} + 2\varepsilon_2), \qquad \varepsilon_1(15 + 6\varepsilon_2\sqrt{5})^{\varepsilon_3/2}, \quad \varepsilon_{1,2,3} = \pm 1.$$
Possible values of the LHS also belong to an explicitly defined finite set: recall that $X = 2\cos\pi n_X/N$, where $n_X = 1, 3, 5, 9, 11$ or 13 for $N = 14$ and $n_X = 1, 5, 7$ or 11 for $N = 12$. Now it is easy to check that the LHS and the RHS of (37) never match, and thus the lengths $N = 12, 14$ are forbidden. □

Proposition 27. *If N is odd and $\omega_Y^2 \neq \omega_Z^2$, then $N \leq 9$.*

Proof. The condition $\omega_Y^2 \neq \omega_Z^2$ guarantees that both p_+ and p_- are non-zero. Assuming for definiteness that n_X is odd, one finds from (35)
$$Y_k - Z_{k+(N-1)/2} = Y_{k'} - Z_{k'+(N-1)/2} \neq 0.$$

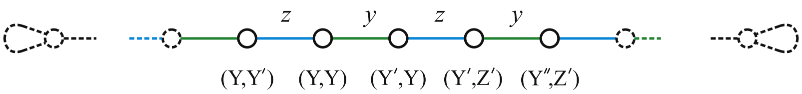


Fig. 3. Line of odd length, $\omega_Y = \omega_Z$.

We can now use the same approach as in the previous proof. One difference is that here we maximize the number of *ordered* couples $\left(Y_k, Z_{k+(N-1)/2}\right)$ of the form (32) characterized by the same value of $Y_k - Z_{k+(N-1)/2}$. This maximal number is equal to 6 (twice the maximal N^c), therefore by Lemma 23 simple cycles of length $N \geq 7$ and the lines of length $N \geq 15$ are forbidden.

The lengths $N = 11, 13$ are excluded similarly to the above, since in this case Y- and Z-coordinates of good points take only a finite number of explicitly defined values. Straightforward computation shows that possible values of X determined from (20) never match $X = 2\cos \pi n_X/N$. $\square$

Remark 28. In the proof of Proposition 27 we used only that $p_- \neq 0$. Therefore the bound "odd $N \leq 9$" also holds for $\omega_Y = \omega_Z \neq 0$ when n_X is even and for $\omega_Y = -\omega_Z \neq 0$ when n_X is odd.

Next we study the case $\omega_Y = \omega_Z$, n_X odd, where the relation (35) gives just $Y_k = Z_{k+(N-1)/2}$. For $\omega_Y = -\omega_Z$, n_X even the upper bound for N is the same by symmetry; recall that e.g. the transformation $\omega_X \mapsto -\omega_X$, $\omega_Y \mapsto -\omega_Y$, $(X, Y, Z) \mapsto (-X, -Y, Z)$ for all $(X, Y, Z) \in O$ yields an orbit equivalent to O.

Proposition 29. *Let N and n_X be odd and let $\omega_Y = \omega_Z \neq 0$. If the graph $\Sigma(O_{yz})$ is a line, then the only possible values of N are* $3, 5, 7, 9, 11, 15$ *and 21.*

Proof. The suborbit graph for odd N is presented in Fig. 3. Each vertex is labeled by its coordinates (Y, Z). For $\omega_Y = \omega_Z$ one has $p_- = 0$, hence (35) implies in particular that for the center point $Z = Y$.

Let us denote $\omega_Y = \omega_Z = \omega$ and $X = 2\cos \pi r_X$, $Y = 2\cos \pi r_Y$, $Y' = 2\cos \pi r_{Y'}$ etc. From the relations

$$Y + Y' + XY = \omega = Y + Z' + XY'$$

one finds an equation of type (24) with $n = 6$:

$$\cos \pi r_{Y'} + \cos \pi (r_X - r_Y) + \cos \pi (r_X + r_Y) = \cos \pi r_{Z'} + \cos \pi (r_X - r_{Y'}) + \cos \pi (r_X + r_{Y'}). \tag{38}$$

We assume that $N \geq 7$, then $r_{X,Y,Y',Z'} \in \mathbb{Q}$ by Lemma 17.

General idea of the proof is to obtain the restrictions on r_X from Lemma 20. Not all solutions listed in Lemma 20 are of interest here because the arguments of cosines in (38) are not all independent. Five entries in the solution 6-tuple, say $\varphi_1 \dots \varphi_5$, should satisfy

(a) $\varepsilon_1\varphi_1 + \varepsilon_2\varphi_2 + \varepsilon_3\varphi_3 + \varepsilon_4\varphi_4 \in \mathbb{Z}$ for some choice of $\varepsilon_{1,2,3,4} = \pm 1$.

(b)$\varepsilon_3\varphi_3 - \varepsilon_4\varphi_4 = 2\varepsilon_5\varphi_5 \pmod{\mathbb{Z}}$ for the same $\varepsilon_{3,4}$ and some $\varepsilon_5 = \pm 1$.

Remark 30. In many cases below, the number of possible solutions for r_X is rather large and their complete description becomes too cumbersome. However, since $r_X = n_X/N$ and N is odd, in practice it is easy to determine admissible values of N by simply looking at odd integers that can appear in the denominator of r_X. The reader should keep in mind that probably not all such admissible values do actually occur. For clarity, the values $N = 3, 5$ (not satisfying the above assumption $N \geq 7$) will not be omitted in the course of this shortcut computation.

First assume that the solution of (38) is equivalent to one of the 6-tuples (VI_1)–(VI_φ):

(VI_1) In this 6-tuple, $1/6$ clearly corresponds to $r_{Z'}$ in (38), otherwise conditions (a) and (b) cannot be simultaneously satisfied. Hence the only possible odd denominator of r_X is 11.

(VI_2) Considering the sum and the difference of any two elements in (VI_2), one readily concludes that the only possible odd denominators of r_X are 3, 7 and 21.

(VI_3) Condition (a) fails unless $1/10$ and $3/10$ correspond to $r_{Y'}$ and $r_{Z'}$ or vice versa. In both cases, however, (b) is violated.

(VI_4) Possible N are 3, 7, 21 by the same argument as in (VI_2).

(VI_5) With the second and the third 6-tuple condition (a) always fails. With the first 6-tuple it can be satisfied only if $1/5$ and $2/5$ correspond to $r_{Y'}$ and $r_{Z'}$ or vice versa, but then (b) is violated.

(VI_φ) Taking the sum and the difference of any two elements (meant to be $r_X \pm r_{Y'}$) we see that odd divisors of the denominator of either r_X or $r_{Y'}$ can only be 3, 5, 15. However, in the second case φ becomes fixed so that admissible N are again 3, 5, 15.

Reducible 6-tuples consisting of one 5-tuple from (V_1)–(V_φ) and one zero cosine (we will say that the solution is of type "$\mathrm{V}_{1,2,3,\varphi}+\mathrm{I}$") can be treated in a completely similar manner, leading to $N = 3, 5, 7, 15, 21$. These values of N are also the only admissible ones for the solutions of type "$\mathrm{IV} + \mathrm{II}_\varphi$", where the solution 6-tuple splits into one of the irreducible quadruples (IV) and a pair of the form (II_φ). Solutions of type "$\mathrm{III}_1 + \mathrm{III}_1$" and "$\mathrm{III}_1 + \mathrm{II}_\varphi + \mathrm{I}$" lead to $N = 3, 5, 15$, and those of type

"$\mathrm{III}_1+\mathrm{III}_\varphi$" to $N=3,5,9,15$. There remain three types of possible rational solution 6-tuples:

(1) "$\mathrm{III}_\varphi+\mathrm{II}_\psi+\mathrm{I}$";

(2) "$\mathrm{II}_\varphi+\mathrm{II}_\psi+\mathrm{II}_\mu$";

(3) "$\mathrm{III}_\varphi+\mathrm{III}_\psi$".

Case (1). We first study the case when (38) contains at least one zero cosine (in particular, this includes (1)). There are four inequivalent possibilities:

(1.1) Set $Y'=0$, then from (38) follows $XY=Z'$. This equation clearly reduces to (24) with $n=3$ and $\varphi_{1,2,3}\in\mathbb{Q}$, hence its solutions are described by Lemma 20. Solutions equivalent to (III_1) can lead only to $N=3,5,15$, and it remains to consider solutions of type "III_φ" and "$\mathrm{II}_\varphi+\mathrm{I}$".

(1.1.1) Solution of $XY=Z'$ has the form (III_φ) only if $X=\pm1$ (i.e. $N=3$) or $Y=\pm1$. In the latter case $Z'=\pm X$ and $\omega=\pm(1+X)$. Now computing $Y''=\omega-Y'-XZ'$ we find $\cos\pi r_{Y''}=\pm(\cos\pi r_X-\cos 2\pi r_X-\cos\pi/3)$. By virtue of Lemma 17, for $N\geq 9$ one has $r_{Y''}\in\mathbb{Q}$. We can thus apply Lemma 20 to the last relation. Irreducible quadruples (IV) lead to $N=3,5,7,15$, solutions of type "$\mathrm{III}_1+\mathrm{I}$" to $N=5$, and solutions of type "$\mathrm{III}_\varphi+\mathrm{I}$" and "$\mathrm{II}_\varphi+\mathrm{II}_\psi$" to $N=3$.

(1.1.2) Now consider solutions of $XY=Z'$ containing at least one zero cosine. Note that $Z'\neq 0$ for $N\geq 7$, since by (35) $Y_k=Z_{k+(N-1)/2}$ and we have already put $Y'=0$. One can therefore assume that $r_Y=r_X\pm 1/2 \pmod{2\mathbb{Z}}$, $Z'=2\cos\pi(2r_X\pm 1/2)$. Computation of Y'' then gives

$$\cos\pi r_{Y''}=\cos\pi(2r_X\pm 1/2)-\cos\pi(3r_X\pm 1/2). \tag{39}$$

If $N\geq 9$, one can apply to (39) Lemma 20. Solutions (III_1) and (III_φ) can lead only to $N=3,5$ and $N=3,5,15$ correspondingly. Since $Y''\neq 0$, the only possible N for solutions of type "$\mathrm{II}_\varphi+\mathrm{I}$" is 3. As a consequence, from now on we can assume that $Y'\neq 0$.

(1.2) Suppose that $Z'=0$. Here we will use two relations of the form (24). The first one, with $n=5$, is merely (38) with $Z'=0$:

$$Y'+XY=XY'. \tag{40}$$

Recall that we can restrict our attention to solutions of (40) of 2 types: "$\mathrm{II}_\varphi+\mathrm{II}_\psi+\mathrm{I}$" and "$\mathrm{III}_\varphi+\mathrm{II}_\psi$". The second equation, with $n=4$, comes from the computation of Y'',

$$Y''=Y+XY. \tag{41}$$

Assume that $N\geq 9$ to guarantee $r_{Y''}\in\mathbb{Q}$ and consider rational solutions of (41) given by Lemma 20. It is easy to check that the quadruples equivalent to (IV) can only lead to $N=3,5,7,15,21$, while for solutions of type "$\mathrm{III}_1+\mathrm{I}$" one has $N=3,5,15$.

Next we examine solutions of (41) of type "$\mathrm{III}_\varphi+\mathrm{I}$". Since $Y,Y''\neq 0$ it can be assumed that $r_Y=r_X\pm 1/2 \pmod{2\mathbb{Z}}$ and then the triple (III_φ) becomes

$$\cos\pi r_{Y''}=\cos\pi(r_X\pm 1/2)+\cos\pi(2r_X\pm 1/2),$$

giving $N=3,9$. Finally, for solutions of type "$\mathrm{II}_\varphi+\mathrm{II}_\psi$", since $Y\neq Y''$, we may write

$$Y''=2\cos\pi(r_Y+r_X),\qquad Y+2\cos\pi(r_Y-r_X)=0.$$

Second relation implies that $r_X=2r_Y+1 \pmod{2\mathbb{Z}}$ (remember that $X\neq\pm 2$). Substituting this into (40), one finds

$$\cos\pi r_{Y'}-\cos\pi r_Y-\cos 3\pi r_Y+\cos\pi(2r_Y+r_{Y'})+\cos\pi(2r_Y-r_{Y'})=0. \tag{42}$$

(1.2.1) Now consider solutions of (42) of type "$\mathrm{II}_\varphi+\mathrm{II}_\psi+\mathrm{I}$". Note that $Y,Y'\neq 0$. Furthermore $\cos 3\pi r_Y=0$ implies $N=3$, therefore it may be assumed that $\cos\pi(2r_Y-r_{Y'})=0$, i.e. $r_{Y'}=2r_Y\pm 1/2 \pmod{2\mathbb{Z}}$. Then (42) transforms into

$$\cos\pi(2r_Y\pm 1/2)-\cos\pi r_Y-\cos 3\pi r_Y+\cos\pi(4r_Y\pm 1/2)=0.$$

We are looking for rational solutions of the last relation that have type "$\mathrm{II}_\varphi+\mathrm{II}_\psi$", hence the only admissible values of N are 3 and 5.

(1.2.2) Consider a solution of (42) of type "$\mathrm{III}_\varphi+\mathrm{II}_\psi$" and take into account the following comments:

- $\cos\pi r_Y$ and $\cos 3\pi r_Y$ cannot belong simultaneously to (II_ψ) because then the denominators of r_Y and r_X would not have odd divisors. They can neither belong simultaneously to (III_φ) unless $N=3$. Therefore we may assume that $\cos\pi r_Y$ and $\cos 3\pi r_Y$ are divided between (III_φ) and (II_ψ).
- $\cos\pi(2r_Y\pm r_{Y'})$ cannot belong simultaneously to (II_ψ) as there is no enough place. It they are both in (III_φ) then either $N=3$ or $Y'=\pm1$. In the latter case, since Y' belongs to (II_ψ), one can only have $N=3,9$. Hence it may be assumed that $\cos\pi(2r_Y\pm r_{Y'})$ are divided between (III_φ) and (II_ψ), and in particular Y' belongs to (III_φ).

Then we are left with two inequivalent possibilities:

$$\begin{cases}\cos\pi r_{Y'} - \cos\pi r_Y + \cos\pi(2r_Y - r_{Y'}) = 0 & (\mathrm{III}_\varphi)\\ \cos 3\pi r_Y = \cos\pi(2r_Y + r_{Y'}). & (\mathrm{II}_\psi)\end{cases} \tag{1.2.2.1}$$

From the second equation one finds either $Y' = Y$ (forbidden) or $r_{Y'} = -5r_Y \pmod{2\mathbb{Z}}$. But then the first equation transforms into $\cos 5\pi r_Y + \cos 7\pi r_Y - \cos\pi r_Y = 0$, which implies $N = 3, 9$.

$$\begin{cases}\cos\pi r_{Y'} - \cos 3\pi r_Y + \cos\pi(2r_Y + r_{Y'}) = 0 & (\mathrm{III}_\varphi)\\ \cos\pi r_Y = \cos\pi(2r_Y - r_{Y'}). & (\mathrm{II}_\psi)\end{cases} \tag{1.2.2.2}$$

Again from the second equation follows either $Y' = Y$ or $r_{Y'} = 3r_Y \pmod{2\mathbb{Z}}$. In the latter case the substitution into the first equation gives $\cos 5\pi r_Y = 0$, hence the only admissible N is 5.

(1.3) Set $\cos\pi(r_X - r_Y) = 0$. This implies $r_Y = r_X + \varepsilon_1/2 \pmod{2\mathbb{Z}}$, $\varepsilon_1 = \pm 1$ and our initial equation (38) transforms into

$$\cos\pi r_{Y'} + \cos\pi(2r_X + \varepsilon_1/2) = \cos\pi r_{Z'} + \cos\pi(r_X - r_{Y'}) + \cos\pi(r_X + r_{Y'}). \tag{43}$$

(1.3.1) We first study solutions of (43) of type "$\mathrm{II}_\varphi + \mathrm{II}_\psi + \mathrm{I}$". All cases when $Y' = 0$ or $Z' = 0$ have been considered above. Moreover $\cos\pi(2r_X + \varepsilon_1/2) = 0$ would lead only to even N, therefore it can be assumed that $\cos\pi(r_X - r_{Y'}) = 0$, i.e. $r_{Y'} = r_X + \varepsilon_2/2 \pmod{2\mathbb{Z}}$, $\varepsilon_2 = \pm 1$. Now $Y \neq Y'$ implies that $\varepsilon_2 = -\varepsilon_1$. Setting e.g. $r_Y = r_X + 1/2$, $r_{Y'} = r_X - 1/2$ in (43) one finds

$$\cos\pi(r_X - 1/2) + \cos\pi(2r_X + 1/2) = \cos\pi r_{Z'} + \cos\pi(2r_X - 1/2).$$

Since we are looking for solutions of type "$\mathrm{II}_\varphi + \mathrm{II}_\psi$" of this equation and since $Y' \neq Z'$, the only possible N is equal to 3.

(1.3.2) Next consider solutions of type "$\mathrm{III}_\varphi + \mathrm{II}_\psi$". It can be assumed that $\cos\pi(r_X \pm r_{Y'})$ do not belong simultaneously to (II_ψ), as this would lead to $X = 0$ ($N = 2$) or $Y' = 0$ (case studied above).

We may further assume that they are not simultaneously in (III_φ), because one would then have $N = 3$ or $Y' = \varepsilon_2$, where $\varepsilon_2 = \pm 1$. In the latter case (43) would transform into

$$\varepsilon_2 \cos\pi/3 + \cos\pi(2r_X + \varepsilon_1/2) = \cos\pi r_{Z'} + \varepsilon_2\cos\pi r_X.$$

Since solutions of this equation should have type "$\mathrm{II}_\varphi + \mathrm{II}_\psi$" and since $Y' \neq Z'$, one concludes that $N = 3$.

(1.3.2.1) Let $\cos\pi r_{Y'}$ be in (II_ψ), then we may write (43) as

$$\begin{cases}\cos\pi(2r_X + \varepsilon_1/2) = \cos\pi r_{Z'} + \cos\pi(r_X + r_{Y'}), & (\mathrm{III}_\varphi)\\ \cos\pi r_{Y'} = \cos\pi(r_X - r_{Y'}). & (\mathrm{II}_\psi)\end{cases}$$

The second equation implies that $r_X = 2r_{Y'} \pmod{2\mathbb{Z}}$. Substituting this into the first equation one finds $\cos\pi(4r_{Y'} + \varepsilon_1/2) = \cos\pi r_{Z'} + \cos 3\pi r_{Y'}$, therefore N can only be equal to 3, 7, 21.

(1.3.2.2) Let $\cos\pi r_{Y'}$ be in (III_φ) and let $\cos\pi(2r_X + \varepsilon_1/2)$ be in (II_ψ). Then one can write

$$\begin{cases}\cos\pi r_{Y'} = \cos\pi r_{Z'} + \cos\pi(r_X - r_{Y'}), & (\mathrm{III}_\varphi)\\ \cos\pi(2r_X + \varepsilon_1/2) = \cos\pi(r_X + r_{Y'}), & (\mathrm{II}_\psi)\end{cases}$$

and it follows that possible values of N are 3, 7, 21. Similarly if both $\cos\pi r_{Y'}$ and $\cos\pi(2r_X + \varepsilon_1/2)$ are in (III_φ), one finds $N = 3, 5, 9, 15$.

(1.4) Finally suppose that $\cos\pi(r_X - r_{Y'}) = 0$. Then $r_{Y'} = r_X + \varepsilon_1/2 \pmod{2\mathbb{Z}}$, $\varepsilon_1 = \pm 1$ and from (38) follows the relation

$$\cos\pi(r_X + \varepsilon_1/2) + \cos\pi(r_X - r_Y) + \cos\pi(r_X + r_Y) = \cos\pi r_{Z'} + \cos\pi(2r_X + \varepsilon_1/2). \tag{44}$$

It is not necessary to examine solutions of (44) of type "$\mathrm{II}_\varphi + \mathrm{II}_\psi + \mathrm{I}$" because all cases when $Y' = 0$, $Z' = 0$ or $\cos\pi(r_X \pm r_Y) = 0$ have already been considered above, and $\cos\pi(2r_X \pm 1/2) = 0$ gives $N = 2$. Hence we may restrict our attention to solutions of type "$\mathrm{III}_\varphi + \mathrm{II}_\psi$".

- $\cos\pi(r_X + \varepsilon_1/2)$ and $\cos\pi(2r_X + \varepsilon_1/2)$ cannot be simultaneously in (II_ψ) unless $N = 3$ and in (III_φ) unless $N = 3, 9$. Therefore one can assume that they are divided between (III_φ) and (II_ψ).
- If both $\cos\pi(r_X \pm r_Y)$ belong to (III_φ), then either $N = 3$ or $Y = \varepsilon_2$, $\varepsilon_2 = \pm 1$, but in the latter case (44) becomes

  $$\cos\pi(r_X + \varepsilon_1/2) + \varepsilon_2\cos\pi r_X = \cos\pi r_{Z'} + \cos\pi(2r_X + \varepsilon_1/2).$$

  The solution of this equation should be of type "$\mathrm{II}_\varphi + \mathrm{II}_\psi$". Since $Y' \neq Z'$ and by the above assumption $\cos\pi(r_X + \varepsilon_1/2)$ and $\cos\pi(2r_X + \varepsilon_1/2)$ are not in the same pair, this can happen only if $\cos\pi(r_X + \varepsilon_1/2) + \varepsilon_2\cos\pi r_X = 0$, i.e. odd N are impossible. Thus we can assume that $\cos\pi(r_X \pm r_Y)$ in (44) are also divided between (III_φ) and (II_ψ) and in particular $\cos\pi r_{Z'}$ belongs to (III_φ).

We then have two inequivalent possibilities:

$$\begin{cases}\cos\pi(r_X+r_Y)=\cos\pi r_{Z'}+\cos\pi(2r_X+\varepsilon_1/2), & (\mathrm{III}_\varphi)\\ \cos\pi(r_X+\varepsilon_1/2)+\cos\pi(r_X-r_Y)=0. & (\mathrm{II}_\psi)\end{cases} \tag{1.4.1}$$

From the second equation one finds that either $Y=0$ or $r_Y=2r_X+\varepsilon_1/2+1 \pmod{2\mathbb{Z}}$. In the former case, substitution into the first equation gives admissible values $N=3,9$, while for the latter $N=3,5,15$.

$$\begin{cases}\cos\pi(r_X+\varepsilon_1/2)+\cos\pi(r_X+r_Y)=\cos\pi r_{Z'}, & (\mathrm{III}_\varphi)\\ \cos\pi(r_X-r_Y)=\cos\pi(2r_X+\varepsilon_1/2). & (\mathrm{II}_\psi)\end{cases} \tag{1.4.2}$$

Here from (II_ψ) follows that either $r_Y=-r_X-\varepsilon_1/2 \pmod{2\mathbb{Z}}$ (forbidden because then $Y=Y'$) or $r_Y=3r_X+\varepsilon_1/2 \pmod{2\mathbb{Z}}$. In the latter case first equation implies that $N=3,5,9,15$.

Case (2). Now we come back to the initial equation (38) and consider its solutions of type "$\mathrm{II}_\varphi+\mathrm{II}_\psi+\mathrm{II}_\mu$".

It can be assumed that $\cos\pi(r_X\pm r_{Y'})$ are not in the same pair, as otherwise $X=0$ ($N=2$) or $Y'=0$ (already considered). Similarly, if both $\cos\pi(r_X\pm r_Y)$ are in the same pair, then $Y=0$ and one can write

$$\begin{cases}\cos\pi r_{Y'}=\cos\pi(r_X-r_{Y'}), & (\mathrm{II}_\varphi)\\ \cos\pi r_{Z'}+\cos\pi(r_X+r_{Y'})=0. & (\mathrm{II}_\psi)\end{cases}$$

Since $X\neq\pm2$, from (II_φ) follows that $r_X=2r_{Y'} \pmod{2\mathbb{Z}}$ and then $Z'=-2\cos3\pi r_{Y'}$. Moreover $Y=0$ implies that $\omega=Y'$, therefore $Y''=-XZ'$, i.e.

$$\cos\pi r_{Y''}=\cos\pi r_{Y'}+\cos5\pi r_{Y'}.$$

For $N\geq9$ we can apply Lemma 20 to the last relation. Its solutions of type (III_1) and (III_φ) lead to $N=3,5$ and $N=3,9$ correspondingly. Since $Y',Y''\neq0$ (because we already have $Y=0$), solutions of type "$\mathrm{II}_\varphi+\mathrm{I}$" are possible only if $N=5$.

Hence we can assume that $\cos\pi(r_X\pm r_Y)$ are divided between two different pairs. These cannot be the same as for $\cos\pi(r_X\pm r_{Y'})$, otherwise the third pair would give $Y'=Z'$. Therefore we may assume one of the pairs in (38) to be

$$\cos\pi(r_X-r_Y)=\cos\pi(r_X-r_{Y'}). \quad (\mathrm{II}_\varphi)$$

Since $Y\neq Y'$, the last relation gives $r_Y=2r_X-r_{Y'} \pmod{2\mathbb{Z}}$. Now for the remaining two pairs there are two inequivalent possibilities:

(2.1) If $\cos\pi r_{Y'}$ and $\cos\pi(r_X+r_Y)$ are in the same pair, then

$$\begin{cases}\cos\pi r_{Y'}+\cos\pi(3r_X-r_{Y'})=0, & (\mathrm{II}_\psi)\\ \cos\pi r_{Z'}+\cos\pi(r_X+r_{Y'})=0. & (\mathrm{II}_\mu)\end{cases}$$

From (II_ψ) one finds that either $N=3$ or $\cos\pi(3r_X-2r_{Y'})/2=0$. In the latter case, compute ω:

$$\omega=Y+Y'+XY=4\cos\pi r_X/2\cos\pi(3r_X-2r_{Y'})/2=0,$$

i.e. the initial assumption $\omega\neq0$ does not hold.

(2.2) If $\cos\pi r_{Y'}$ and $\cos\pi(r_X+r_{Y'})$ are in the same pair, then

$$\begin{cases}\cos\pi r_{Y'}=\cos\pi(r_X+r_{Y'}), & (\mathrm{II}_\psi)\\ \cos\pi r_{Z'}=\cos\pi(3r_X-r_{Y'}). & (\mathrm{II}_\mu)\end{cases}$$

First equation implies that $r_X=-2r_{Y'} \pmod{2\mathbb{Z}}$. Therefore $X=2\cos2\pi r_{Y'}$, $Y=2\cos5\pi r_{Y'}$, $Z'=2\cos7\pi r_{Y'}$. Let us compute $\omega=Y+Y'+XY$:

$$\omega=2\cos\pi r_{Y'}+2\cos3\pi r_{Y'}+2\cos5\pi r_{Y'}+2\cos7\pi r_{Y'}.$$

The computation of $Y''=\omega-Y'-XZ'$ now gives

$$\cos\pi r_{Y''}=\cos3\pi r_{Y'}+\cos7\pi r_{Y'}-\cos9\pi r_{Y'}. \tag{45}$$

For $N\geq9$, we can apply to (45) Lemma 20. Solutions of type (IV), "III_1+I" and "III_φ+I" can lead only to $N=3,5,7,9,15,21$. Since $Y''\neq Z'$, solutions of type "$\mathrm{II}_\varphi+\mathrm{II}_\psi$" are possible only if $N=5$.

Case(3). It remains to consider solutions of (38) of type "$\mathrm{III}_\varphi+\mathrm{III}_\psi$".

(3.1) If both $\cos\pi(r_X\pm r_{Y'})$ appear in the same triple, then $N=3$ or $Y'=\pm1$. In the latter case, (38) transforms into

$$\pm\cos\pi/3+\cos\pi(r_X+r_Y)+\cos\pi(r_X-r_Y)=\cos\pi r_{Z'}\pm\cos\pi r_X. \tag{46}$$

The solution of (46) should have type "$\mathrm{III}_\varphi+\mathrm{II}_\psi$", and moreover $\cos\pi r_X$ belongs to (II_ψ). If the second cosine in (II_ψ) is $\cos\pi/3$, then $N=3$. If $\cos\pi r_{Z'}\pm\cos\pi r_X=0$, then from (III_φ) again follows $N=3$. Therefore it can be assumed that

$$\begin{cases}\pm\cos\pi/3+\cos\pi(r_X+r_Y)=\cos\pi r_{Z'}, & (\mathrm{III}_\varphi)\\ \cos\pi(r_X-r_Y)=\pm\cos\pi r_X. & (\mathrm{II}_\psi)\end{cases}$$

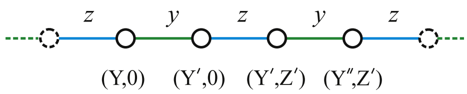


Fig. 4. Labeling of $\Sigma(O_{yz})$ in Lemma 31.

Since $Y \neq \pm 2$, the second equation implies that $r_Y = 2r_X + 1/2 \mp 1/2$, but then from the first equation follows $N = 3, 9$. Hence from now on we assume that $\cos\pi(r_X \pm r_{Y'})$ belong to different triples.

(3.2) If $\cos\pi(r_X \pm r_Y)$ are in the same triple, then $N = 3$ or $Y = \pm 1$. In the latter case (38) can be rewritten as

$$\begin{cases} \cos\pi r_{Y'} = \cos\pi r_{Z'} + \cos\pi(r_X + r_{Y'}), & (\mathrm{III}_\varphi) \\ \pm\cos\pi r_X = \cos\pi(r_X - r_{Y'}). & (\mathrm{II}_\psi) \end{cases}$$

Again from (II_ψ) follows $r_{Y'} = 2r_X + 1/2 \mp 1/2$, and (III_φ) then implies that $N = 3, 5, 15$. Therefore we assume in the following that $\cos\pi(r_X \pm r_Y)$, as well as $\cos\pi r_{Y'}$ and $\cos\pi r_{Z'}$, are divided between the two triples.

(3.3) Without loss of generality we can now write (38) as

$$\begin{cases} \cos\pi r_{Y'} + \cos\pi(r_X - r_Y) - \cos\pi(r_X - r_{Y'}) = 0, & (\mathrm{III}_\varphi) \\ \cos\pi r_{Z'} + \cos\pi(r_X + r_{Y'}) - \cos\pi(r_X + r_Y) = 0, & (\mathrm{III}_\psi) \end{cases} \tag{47}$$

or, in another form,

$$\begin{cases} \cos\pi r_{Y'} + 2\sin\dfrac{\pi(2r_X - r_Y - r_{Y'})}{2}\sin\dfrac{\pi(r_Y - r_{Y'})}{2} = 0, & (\mathrm{III}_\varphi) \\ \cos\pi r_{Z'} + 2\sin\dfrac{\pi(2r_X + r_Y + r_{Y'})}{2}\sin\dfrac{\pi(r_Y - r_{Y'})}{2} = 0. & (\mathrm{III}_\psi) \end{cases}$$

If $\sin\frac{\pi(r_Y - r_{Y'})}{2} \neq \pm\frac{1}{2}$, then one should simultaneously have

$$\sin\frac{\pi(2r_X - r_Y - r_{Y'})}{2} = \frac{\varepsilon_1}{2}, \qquad \sin\frac{\pi(2r_X + r_Y + r_{Y'})}{2} = \frac{\varepsilon_2}{2}, \tag{48}$$

where $\varepsilon_{1,2} = \pm 1$ (in fact $\varepsilon_2 = -\varepsilon_1$, otherwise $Y' = Z'$). Eqs. (48) lead to $N = 3$, therefore we can assume that

$$\sin\frac{\pi(r_Y - r_{Y'})}{2} = \frac{\varepsilon_3}{2}, \quad \varepsilon_3 = \pm 1. \tag{49}$$

Let us compute $Y'' = Y + XY - XZ'$ using (47) and (49). After some simplifications one finds

$$\cos\pi r_{Y''} = \cos\pi(r_X + r_Y) + \cos\pi(r_X - r_{Y'}) + \varepsilon_3\sin\frac{\pi(4r_X + r_Y + r_{Y'})}{2}. \tag{50}$$

Relation (49) implies that $r_Y = r_{Y'} + \varepsilon_3/3 \pmod{4\mathbb{Z}}$ or $r_Y = r_{Y'} + 5\varepsilon_3/3 \pmod{4\mathbb{Z}}$. Similarly, the first relation in (47) gives either $N = 3$ or $r_X = 2r_{Y'} + \varepsilon_4/3 \pmod{2\mathbb{Z}}$, $\varepsilon_4 = \pm 1$. We now substitute this into (50) and apply Lemma 20 (for $N \geq 9$). Solutions of type (IV) and "$\mathrm{III}_1+\mathrm{I}$" then lead to admissible values $N = 3, 5, 7, 9, 15, 21$ and $N = 3, 5, 15$ correspondingly, while solutions of type "$\mathrm{III}_\varphi + \mathrm{I}$" and "$\mathrm{II}_\varphi + \mathrm{II}_\psi$" give $N = 3, 5, 9, 15$ and $N = 3, 9$. This concludes the proof of Proposition 29. □

Lemma 31. *Let N and n_X be odd and let $\omega_Y = \omega_Z \neq 0$. If the graph $\Sigma(O_{yz})$ is a simple cycle and O_{yz} contains a point with coordinate Z (or Y) equal to* 0, *then the only possible values of N are* 3, 5, 7, 9, 15, 21.

Proof. Analogously to the previous proof, let us label the vertices of $\Sigma(O_{yz})$ by their coordinates (Y, Z), as shown in Fig. 4. Because of the simple cycle assumption all points of O_{yz} are good, therefore all $\{Y_k\}$ and $\{Z_k\}$ have the form (32). It will be assumed that $N > 3$, then by Lemma 23 four numbers Y, Y', Y'', Z' are distinct and non-zero (recall that $Y_k = Z_{k+(N-1)/2}$).

We now apply Lemma 20 to the relation

$$\cos\pi r_Y + \cos\pi r_{Y'} = \cos\pi r_{Z'} + \cos\pi(r_X + r_{Y'}) + \cos\pi(r_X - r_{Y'}). \tag{51}$$

Its solutions of type (V_φ), (V_1)–(V_3), "IV + I" can lead only to $N = 3, 5, 7, 15, 21$.

The solutions of type "$\mathrm{II}_\varphi + \mathrm{II}_\psi + \mathrm{I}$" are forbidden. Indeed, since $Y, Y', Z' \neq 0$, in this case one could write $\cos\pi(r_X - r_{Y'}) = 0$, but then one of the pairs (II_φ), (II_ψ) would give $Y = Z'$ or $Y' = Z'$ (impossible) or $Y + Y' = 0$ (excluded because then $\omega = 0$).

Next we consider solutions of type "$\mathrm{III}_1 + \mathrm{II}_\varphi$". Since $Y' \neq 0$, two cosines $\cos\pi(r_X \pm r_{Y'})$ cannot belong both to (II_φ). They can neither be simultaneously in (III_1), as (II_φ) would then give $Y = Z'$ or $Y' = Z'$ or $Y + Y' = 0$. Therefore it can be assumed that $\cos\pi(r_X - r_{Y'})$ belongs to (II_φ) and $\cos\pi(r_X + r_{Y'})$ is in (III_1). Now if $\cos\pi r_{Y'}$ is in (III_1), then admissible values of N are 3, 5, 15. If $\cos\pi r_{Y'}$ belongs to (II_φ), then $r_X = 2r_{Y'} \pmod{2\mathbb{Z}}$. Substituting this into (III_1), we obtain $N = 5, 9, 15$.

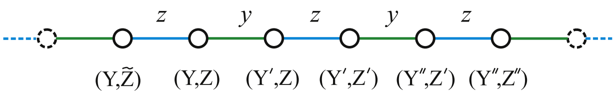


Fig. 5. Labeling of $\Sigma(O_{yz})$ in Proposition 32.

It remains to consider solutions of (51) of type "$\mathrm{III}_\varphi + \mathrm{II}_\psi$". By the same argument as above we can assume that $\cos\pi(r_X - r_{Y'})$ is in (II_ψ) and $\cos\pi(r_X + r_{Y'})$ is in (III_φ).

Assume that $\cos\pi r_{Y'}$ is in (II_ψ). Then $r_X = 2r_{Y'} \pmod{2\mathbb{Z}}$ and the triple (III_φ) becomes

$$\cos\pi r_Y = \cos\pi r_{Z'} + \cos 3\pi r_{Y'}.$$

Therefore we can assume that $r_Y = 3r_{Y'} \pm 1/3 \pmod{2\mathbb{Z}}$, $r_{Z'} = 3r_{Y'} \pm 2/3 \pmod{2\mathbb{Z}}$. Let us substitute these expressions into an easily verified relation

$$\cos\pi r_{Y''} = \cos\pi r_Y - \cos\pi(r_X - r_{Z'}) - \cos\pi(r_X + r_{Z'}). \tag{52}$$

Its solutions of type (IV) and "$\mathrm{III}_1 + \mathrm{I}$" lead to admissible values $N = 3, 5, 7, 15, 21$ and $N = 3, 5, 15$ correspondingly (in fact this conclusion does not depend on any of our previous assumptions). Since $Y, Y'' \neq 0$, solutions of type "$\mathrm{III}_\varphi + \mathrm{I}$" give $N = 3, 5, 15$. Finally, since $Y \neq Y''$, solutions of type "$\mathrm{II}_\varphi + \mathrm{II}_\psi$" are only possible for $N = 3$.

On the other hand, if $\cos\pi r_{Y'}$ belongs to the triple (III_φ) then, since $\cos\pi(r_X + r_{Y'})$ is also in (III_φ), we can set $r_X = -2r_{Y'} + \varepsilon/3 \pmod{2\mathbb{Z}}$, $\varepsilon = \pm 1$, otherwise $N = 3$. Hence

(1) If $\cos\pi r_Y$ is the third cosine in (III_φ), then

$$Y = -2\cos\pi(r_{Y'} + \varepsilon/3), \qquad Z' = -2\cos\pi(3r_{Y'} - \varepsilon/3).$$

Now let us look at Eq. (52). When its solution has type "$\mathrm{III}_\varphi + \mathrm{I}$", it can be assumed that $\cos\pi(r_X - r_{Z'}) = 0$, but then $N = 3, 5, 15$. For solutions of type "$\mathrm{II}_\varphi + \mathrm{II}_\psi$" we can write $\cos\pi r_Y = \cos\pi(r_X - r_{Z'})$, which leads to $N = 3, 9$.

(2) If $\cos\pi r_Y$ belongs to (II_ψ), then one finds

$$Y = 2\cos\pi(3r_{Y'} - \varepsilon/3), \qquad Z' = 2\cos\pi(r_{Y'} + \varepsilon/3).$$

In this case, solutions of (52) of type "$\mathrm{III}_\varphi + \mathrm{I}$" and "$\mathrm{II}_\varphi + \mathrm{II}_\psi$" lead to admissible values $N = 3, 9$. □

Proposition 32. *Let N and n_X be odd and let $\omega_Y = \omega_Z \neq 0$. If the graph $\Sigma(O_{yz})$ is a simple cycle then the only possible values of N are* 3, 5, 7, 9, 11, 15, 21.

Proof. Let us start with the obvious relation $Y + XZ = Y'' + XZ'$ (see Fig. 5), written as

$$\cos\pi r_Y + \cos\pi(r_X + r_Z) + \cos\pi(r_X - r_Z) = \cos\pi r_{Y''} + \cos\pi(r_X + r_{Z'}) + \cos\pi(r_X - r_{Z'}). \tag{53}$$

We can assume that this relation does not contain zero cosines. Indeed, the case when $Y = 0$ or $Y'' = 0$ is completely described by Lemma 31. If $\cos\pi(r_X \pm r_Z) = 0$ or $\cos\pi(r_X \pm r_{Z'}) = 0$, then Z or Z' is equal to $\pm\sqrt{4 - X^2}$. Now recall that by Lemma 23 in a simple cycle all $\{Z_k\}$ are distinct, therefore already for $N \geq 5$ it will be possible to find a pair (Z_k, Z_{k+1}) which does not contain prescribed two values $\pm\sqrt{4 - X^2}$ (Assumption 1).

Next we exclude solutions of type (VI_1)–(VI_5), "$\mathrm{IV} + \mathrm{II}_\varphi$", "$\mathrm{III}_1 + \mathrm{III}_1$", "$\mathrm{III}_1 + \mathrm{III}_\varphi$", as they can lead only to $N = 3, 5, 7, 9, 11, 15, 21$ (note that solutions of (53) satisfy a condition similar to (a) in the proof of Proposition 29). Then there remain three types of possible solution 6-tuples:

(1) "$\mathrm{II}_\varphi + \mathrm{II}_\psi + \mathrm{II}_\mu$";
(2) "$\mathrm{III}_\varphi + \mathrm{III}_\psi$";
(3) "VI_φ".

Case (1). It can be assumed that two cosines $\cos\pi(r_X \pm r_Z)$ (and $\cos\pi(r_X \pm r_{Z'})$) are divided between two different pairs. Otherwise $Z = 0$ (resp. $Z' = 0$) and one obtains restrictions on N from Lemma 31. The pairs cannot be the same in both cases because then $Y = Y''$. Therefore we can set one of the pairs to be

$$\cos\pi(r_X - r_Z) = \cos\pi(r_X - r_{Z'}). \quad (\mathrm{II}_\varphi) \tag{54}$$

Since $Z \neq Z'$, one has $r_{Z'} = 2r_X - r_Z \pmod{2\mathbb{Z}}$. For the remaining two pairs, there are two inequivalent possibilities:

$$\begin{cases} \cos\pi r_Y + \cos\pi(r_X + r_Z) = 0, & (\mathrm{II}_\psi) \\ \cos\pi r_{Y''} + \cos\pi(3r_X - r_Z) = 0. & (\mathrm{II}_\mu) \end{cases} \tag{1.1}$$

Here from $Y + Y' + XZ = Z + Z' + XY'$ follows that either $N = 3$ or $Y' = -2\cos\pi(r_X - r_Z)$. In the latter case, however, computing $\omega = Y + Y' + XZ$ we find forbidden value $\omega = 0$.

$$\begin{cases} \cos\pi r_Y = \cos\pi(3r_X - r_Z), & (\mathrm{II}_\psi) \\ \cos\pi r_{Y''} = \cos\pi(r_X + r_Z). & (\mathrm{II}_\mu) \end{cases} \tag{1.2}$$

Substituting these relations into $\tilde{Z} + XY = Z' + XY'$ and $Z'' + XY'' = Z + XY'$, one obtains

$$\cos \pi r_{\tilde{Z}} + \cos \pi (4r_X - r_Z) \;=\; \cos \pi (r_X - r_{Y'}) + \cos \pi (r_X + r_{Y'}), \tag{55}$$

$$\cos \pi r_{Z''} + \cos \pi (2r_X + r_Z) \;=\; \cos \pi (r_X - r_{Y'}) + \cos \pi (r_X + r_{Y'}). \tag{56}$$

Solutions of (55), (56) of type (IV) and "$\mathrm{III}_1 + \mathrm{I}$" can lead only to $N = 3, 5, 7, 15, 21$, therefore we can restrict our attention to solutions of type "$\mathrm{III}_\varphi + \mathrm{I}$" and "$\mathrm{II}_\varphi + \mathrm{II}_\psi$".

(1.2.1) Suppose that the solution of (55) is of type "$\mathrm{III}_\varphi + \mathrm{I}$". If $\cos \pi (4r_X - r_Z) = 0$ and $\cos \pi (r_X \pm r_{Y'})$ are in (III_φ), then $N = 3$ or $r_Z = 4r_X + \varepsilon_1/2 \pmod{2\mathbb{Z}}$, $Y' = \varepsilon_2$, $\varepsilon_{1,2} = \pm 1$. In the second case (56) transforms into

$$\cos \pi r_{Z''} + \cos \pi (6r_X + \varepsilon_1/2) = \varepsilon_2 \cos \pi r_X.$$

Now if the solution of this equation has type (III_φ) or (III_1), then $N = 3, 5, 7, 15, 21$. Since it can be assumed that $Z'' \neq 0$, type "$\mathrm{II}_\varphi + \mathrm{I}$" solutions give $N = 3$.

On the other hand, if $\cos \pi (r_X - r_{Y'}) = 0$, i.e. $r_{Y'} = r_X + \varepsilon_1/2 \pmod{2\mathbb{Z}}$, $\varepsilon_1 = \pm 1$, then the triple (III_φ) in (55) is given by

$$\cos \pi r_{\tilde{Z}} + \cos \pi (4r_X - r_Z) = \cos \pi (2r_X + \varepsilon_1/2).$$

This relation implies that either (a) $r_Z = 2r_X - \varepsilon_1/2 + \varepsilon_2/3 \pmod{2\mathbb{Z}}$ or (b) $r_Z = 6r_X + \varepsilon_1/2 + \varepsilon_2/3 \pmod{2\mathbb{Z}}$. In the case (a) Eq. (56) transforms into

$$\cos \pi r_{Z''} + \cos \pi (4r_X - \varepsilon_1/2 + \varepsilon_2/3) = \cos \pi (2r_X + \varepsilon_1/2).$$

Its solutions of type (III_φ) and "$\mathrm{II}_\varphi + \mathrm{I}$" lead to admissible values $N = 3, 9$. Similarly, in the case (b) relation (56) gives $N = 3, 5, 9, 15$.

(1.2.2) The case when the solution of (56) is of type "$\mathrm{III}_\varphi + \mathrm{I}$" is treated analogously to (1.2.1), hence we can assume that solutions of both (55) and (56) have the form "$\mathrm{II}_\varphi + \mathrm{II}_\psi$". Thanks to Lemma 31, it can be assumed that $Y' \neq 0$ so that $\cos \pi (r_X \pm r_{Y'})$ in (55), (56) are divided between the two pairs. Since $\tilde{Z} \neq Z''$, we may write without loss of generality

$$\begin{cases} \cos \pi (4r_X - r_Z) = \cos \pi (r_X - r_{Y'}), \\ \cos \pi (2r_X + r_Z) = \cos \pi (r_X + r_{Y'}). \end{cases}$$

From the first equation follows either $r_{Y'} = -3r_X + r_Z \pmod{2\mathbb{Z}}$ (forbidden because then $Y = Y'$) or $r_{Y'} = 5r_X - r_Z \pmod{2\mathbb{Z}}$. In the latter case the second equation becomes

$$\cos \pi (2r_X + r_Z) = \cos \pi (6r_X - r_Z),$$

and implies that $2r_X - r_Z \in \mathbb{Z}$. This in turn gives $Z' = \pm 2$, which is impossible as all points in O_{yz} are good.

Case (2). Suppose that Z and Z' are not equal to ± 1 (Assumption 2). Clearly for $N \geq 9$ one will always be able to find in O_{yz} a pair (Z, Z') satisfying Assumptions 1 and 2. Then in (53) the two cosines $\cos \pi (r_X \pm r_Z)$, as well as $\cos \pi (r_X \pm r_{Z'})$, are divided between the two triples (III_φ) and (III_ψ), otherwise $X = \pm 1$ and $N = 3$. We can therefore write

$$\begin{cases} \cos \pi r_Y + \cos \pi (r_X - r_Z) - \cos \pi (r_X - r_{Z'}) = 0, & (\mathrm{III}_\varphi) \\ \cos \pi r_{Y''} - \cos \pi (r_X + r_Z) + \cos \pi (r_X + r_{Z'}) = 0. & (\mathrm{III}_\psi) \end{cases} \tag{57}$$

Similarly to the proof of Proposition 29, case (3.3) one can show that

$$\sin \frac{\pi (r_Z - r_{Z'})}{2} = \pm \frac{1}{2},$$

i.e. $r_{Z'} = r_Z + \varepsilon_1/3 \pmod{2\mathbb{Z}}$, $\varepsilon_1 = \pm 1$.

From $\omega = Y + Y' + XZ = Z + Z' + XY'$ follows that

$$(X - 1)\omega = XY + (X^2 - 2)Z + Z - Z'.$$

Substituting (57) into this relation, we find

$$(X - 1)\omega = 2 \cos \pi (2r_X + r_Z) + 2 \cos \pi (2r_X - r_{Z'}) = 2 \cos \pi (2r_X + r_Z) + 2 \cos \pi (2r_X - r_Z - \varepsilon_1/3).$$

Recall that for a simple cycle of length N, one may write N relations of the form (53) which correspond to different unordered pairs (Z, Z'). Suppose there exists a second relation whose solution has the form "$\mathrm{III}_\varphi + \mathrm{III}_\psi$", and the associated pair $(\bar{Z}, \bar{Z}')$ satisfies Assumptions 1 and 2. Then we can write

$$\cos \pi (2r_X + r_Z) + \cos \pi (2r_X - r_Z - \varepsilon_1/3) = \cos \pi (2r_X + r_{\bar{Z}}) + \cos \pi (2r_X - r_{\bar{Z}} - \varepsilon_2/3), \tag{58}$$

where $r_{\bar{Z}'} = r_{\bar{Z}} + \varepsilon_2/3 \pmod{2\mathbb{Z}}$, $\varepsilon_2 = \pm 1$. If $\varepsilon_1 = \varepsilon_2$, then (58) implies that either $N = 3$ or the pairs (Z, Z') and $(\bar{Z}, \bar{Z}')$ coincide. Let us now set $\varepsilon_2 = -\varepsilon_1$ and consider rational solutions of (58).

Solutions of type (IV) and "$\mathrm{III}_1+\mathrm{I}$" can lead only to $N=3,5,7,15,21$ and $N=3,5,15$ correspondingly. Solutions of type "$\mathrm{III}_\varphi+\mathrm{I}$" give $N=3,9$. Finally, since $\omega\neq 0$ and it may be assumed that $X\neq 1$, for solutions of type "$\mathrm{II}_\varphi+\mathrm{II}_\psi$" there are two possibilities:

$$\begin{cases}\cos\pi(2r_X+r_Z)=\cos\pi(2r_X+r_{\tilde Z}),\\ \cos\pi(2r_X-r_Z-\varepsilon_1/3)=\cos\pi(2r_X-r_{\tilde Z}+\varepsilon_1/3).\end{cases}\tag{2.1}$$

If $r_Z=r_{\tilde Z}\pmod{2\mathbb{Z}}$, then the second equation implies that $r_Z=2r_X+(1-\varepsilon_3)/2\pmod{2\mathbb{Z}}$, $\varepsilon_3=\pm1$. Assume that $N\neq 3$, then from the relation $(X-1)(Y'-Z)=Y-Z'$ we find

$$\cos\pi r_{Y'}=\varepsilon_3\Big(\cos 2\pi r_X-\cos\pi(r_X+\varepsilon_1/3)-\cos\pi/3\Big).$$

Rational solutions of this equation lead to admissible values $N=3,5,7,9,15$. Now if we take as the solution of the first equation in (2.1) $r_{\tilde Z}=-4r_X-r_Z\pmod{2\mathbb{Z}}$, then from the second equation follows $r_Z=-2r_X-\varepsilon_1/3+(1-\varepsilon_3)/2\pmod{2\mathbb{Z}}$. Computing Y' from $(X-1)(Y'-Z')=Y''-Z$, one finds the same values of N.

$$\begin{cases}\cos\pi(2r_X+r_Z)=\cos\pi(2r_X-r_{\tilde Z}+\varepsilon_1/3),\\ \cos\pi(2r_X-r_Z-\varepsilon_1/3)=\cos\pi(2r_X+r_{\tilde Z}).\end{cases}\tag{2.2}$$

This case is completely analogous to (2.1).

Case(3). Recall that solutions of (24) relevant for (53) should satisfy an additional constraint $\varepsilon_1\varphi_1+\varepsilon_2\varphi_2+\varepsilon_3\varphi_3+\varepsilon_4\varphi_4\in\mathbb{Z}$ with some $\varepsilon_{1,2,3,4}=\pm1$. This condition implies that $\varphi\pm1/6$ in (VI_φ) belong or do not belong to $\{\varphi_1,\varphi_2,\varphi_3,\varphi_4\}$ simultaneously, otherwise admissible N are 3, 5, 15. Furthermore if we assume that $N\neq 3,5,15$, the unordered pairs (r_X+r_Z,r_X-r_Z) and $(r_X+r_{Z'},r_X-r_{Z'})$ can only be equivalent to the following:

(3.1) $(2\varphi+1/3,2\varphi-1/3)$ and $(2\varphi+3/5,2\varphi-3/5)$,

(3.2) $(2\varphi+1/3,2\varphi-1/3)$ and $(2\varphi+1/5,2\varphi-1/5)$,

(3.3) $(2\varphi+1/5,2\varphi-1/5)$ and $(2\varphi+2/5,2\varphi-2/5)$.

Here $\varphi\in\mathbb{Q}$ and all entries in (3.1)–(3.3) are considered mod $2\mathbb{Z}$. Now observe that in (3.1) and (3.2) either Z or Z' is equal to ±1, therefore such 6-tuples can be excluded by Assumption 1. In the case (3.3), unordered pair (Z,Z') is equal to $(2\cos\pi/5,2\cos2\pi/5)$ or $(-2\cos\pi/5,-2\cos2\pi/5)$.

Let us now summarize the above results. If $N\neq 3,5,7,9,11,15,21$, then N relations (53) can have only the following solutions:

(a) with Z or Z' equal to $\pm1,\pm\sqrt{4-X^2}$,

(b) solutions of type "$\mathrm{III}_\varphi+\mathrm{III}_\psi$" (and "$\mathrm{VI}_\varphi$") satisfying Assumptions 1 and 2; these appear in O_{yz} at most once (resp. twice).

However, under such restrictions the length of O_{yz} cannot exceed 11 because of Lemma 23 (as all Z_k in the simple cycle are distinct). □

Proposition 33. *Let $\omega_Y=\omega_Z=0$. Then either $N\leq 15$ or the suborbit O_{yz} has the form*

$$\begin{cases}X=2\cos\pi r_X,\\ Y_k=-2\cos\pi\big[r_X(1+2k_0-2k)+r_Z\big],\\ Z_k=2\cos\pi\big[2r_X(k_0-k)+r_Z\big].\end{cases}\tag{59}$$

where $k_0\in\{0,1,\ldots,N-1\}$ and $r_{X,Z}\in\mathbb{Q}$.

Proof. Let us consider the relation (see Fig. 5)

$$\cos\pi r_Y+\cos\pi r_{Y'}+\cos\pi(r_X+r_Z)+\cos\pi(r_X-r_Z)=0.\tag{60}$$

For $N\geq 15$ ($N\geq 6$ in the simple cycle case) one will always be able to find in O_{yz} a solution with $r_{Y,Y',Z}\in\mathbb{Q}$ satisfying the restrictions $Y,Y'\neq 0$ and $Z\neq 0,\pm\sqrt{4-X^2}$. With these requirements, the solution of (60) cannot be of type "$\mathrm{III}_1+\mathrm{I}$" or "$\mathrm{III}_\varphi+\mathrm{I}$" as the relation (60) does not contain zero cosines. Moreover one cannot have solutions of type "$\mathrm{II}_\varphi+\mathrm{II}_\psi$" with $Y+Y'=0$ unless $X=0$, i.e. $N=2$.

For the remaining "$\mathrm{II}_\varphi+\mathrm{II}_\psi$" solutions one can write

$$Y=-2\cos\pi(r_X+r_Z),\qquad Y'=-2\cos\pi(r_X-r_Z).$$

Setting $Y=Y_{k_0}$, $Y'=Y_{k_0+1}$ we find that α,β in (21) are given by

$$\alpha=-2\cos\pi\big[r_X(1+2k_0)+r_Z\big],\qquad \beta=2\cos\pi\big[2k_0r_X+r_Z\big],$$

and hence $\{Y_k\}$, $\{Z_k\}$ have the form (59).

Now we can assume that all solutions satisfying the above restrictions are equivalent to the quadruples (IV). This leads to admissible values $N=3,5,7,15,30,42$. However, the lengths $N=30,42$ can be excluded because it is not possible to generate from (IV) a sufficient number of solutions with the same value of X and different Z.

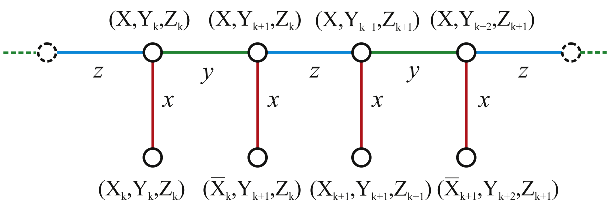


Fig. 6. Labeling of $\Sigma(O_{yz})$ in Proposition 34.

Example. Checking all the quadruples (IV) with $X = 2\cos\pi/30$ we find that there are only six possible values of Z: $\pm 2\cos 7\pi/30$, $\pm 2\cos 11\pi/30$ and $\pm 2\cos 13\pi/30$. $\square$

Assume that O_{yz} has the form (59). If $\omega_X = 0$, then from (10) and (59) follows that $\omega_4 = 0$. Finite orbits of the induced $\bar{\Lambda}$ action (14) with $\omega_X = \omega_Y = \omega_Z = \omega_4 = 0$ will be called *Cayley orbits* because in this case the Jimbo–Fricke relation (10) reduces to Cayley cubic

$$XYZ + X^2 + Y^2 + Z^2 - 4 = 0. \tag{61}$$

Cayley orbits admit a simple characterization, though their size can be arbitrarily large. To each of these orbits one can assign in a non-unique way a pair of rational numbers. Indeed, consider an arbitrary point $\mathbf{r} = (X, Y, Z) \in O$. It is not fixed by at least one transformation, say x (we assume that O consists of more than one point). Lemma 17 then implies that $Y = 2\cos\pi r_Y, Z = 2\cos\pi r_Z$ with $r_{Y,Z} \in \mathbb{Q}$. The relation (61) can be rewritten as

$$\big(X + 2\cos\pi(r_Y + r_Z)\big)\big(X + 2\cos\pi(r_Y - r_Z)\big) = 0,$$

hence we may assume that $X = -2\cos\pi(r_Y + r_Z)$ (if $X = -2\cos\pi(r_Y - r_Z)$, start from $x(X, Y, Z)$). Now making one step from (X, Y, Z) by x, y and z one finds

$$\begin{cases} X(x(\mathbf{r})) = -2\cos\pi(r_Y - r_Z), \\ Y(y(\mathbf{r})) = 2\cos\pi(r_Y + 2r_Z), \\ Z(z(\mathbf{r})) = 2\cos\pi(2r_Y + r_Z). \end{cases}$$

Continuing by induction we see that for any other point $(X', Y', Z') \in O$ one has $X' = 2\cos\pi r_{X'}$, $Y' = 2\cos\pi r_{Y'}$, $Z' = 2\cos\pi r_{Z'}$, where $r_{X',Y',Z'} \in \mathbb{Q}$ and the denominators of $r_{X',Y',Z'}$ are divisors of the common denominator of r_Y and r_Z. Lemma 23 then guarantees that O is finite.

Proposition 34. *Let $\omega_Y = \omega_Z = 0$. If O_{yz} has the form* (59) *and $\omega_X \neq 0$, then $N \leq 12$.*

Proof. Let us make one step by x from each point of O_{yz} (see Fig. 6). Using (59), from the relations $\omega_X = X + X_k + Y_k Z_k = X + \bar{X}_k + Y_{k+1} Z_k$ one finds

$$\begin{aligned} \omega_X &= X_k - 2\cos\pi\big[r_X(4k_0 - 4k + 1) + 2r_Z\big] \\ &= \bar{X}_k - 2\cos\pi\big[r_X(4k_0 - 4k - 1) + 2r_Z\big], \end{aligned} \tag{62}$$

for any $k = 0, 1, \ldots, N - 1$. If the point (X_k, Y_k, Z_k) is good then by Lemma 17

$$X_k = 2\cos\pi r_{X_k}, \quad r_{X_k} \in \mathbb{Q}. \tag{63}$$

It can be bad in two cases:

(1) The graph of O_{yz} is a line, (X, Y_k, Z_k) corresponds to one of its end vertices and $X_k = X$. Since $N > 1$, X_k still has the form (63).

(2) (X_k, Y_k, Z_k) is fixed by the transformations y and z. Then from (20) follows that either $X_k = \pm 2$ or $Y_k = Z_k = 0$. In the latter case, however, the condition $N > 1$ is violated since the whole orbit O consists of only two points $(X, 0, 0)$ and $(\omega_X - X, 0, 0)$.

Thus all X_k and $\bar{X}_k$ have the form (63) and the solutions of (62) are classified by Lemma 20.

Introduce $2N$ quantities $W_0, \ldots, W_{2N-1}$ defined by

$$W_{2k} = X_k - \omega_X, \qquad W_{2k+1} = \bar{X}_k - \omega_X, \quad k = 0, \ldots, N - 1.$$

Obviously, $W_l = 2\cos\pi\big[r_X(1 + 4k_0 - 2l) + 2r_Z\big]$. We now want to show that the number of coinciding W_l cannot exceed 4. Indeed, fix some l, then $W_{l'} = W_l$ implies that (a) $l' - l = 0 \bmod N$ or (b) $r_X(1 + 4k_0 - l - l') + 2r_Z \in \mathbb{Z}$. The former case leads to one compatible $W_{l'}$, while the latter gives at most two: if l'_1 and l'_2 satisfy (b), then necessarily $l'_1 - l'_2 = 0 \bmod N$.

In the proof of Propositions 26 and 27 we have shown that the maximal number of ordered pairs $(\cos\pi r_1, \cos\pi r_2)$, $r_{1,2} \in \mathbb{Q}$ such that $\cos\pi r_1 + \cos\pi r_2 = \text{const} \neq 0$ is equal to 6. Hence the number of distinct possible values for all W_l's cannot exceed 6 and the total number of W_l's, equal to $2N$, cannot exceed 24. $\square$

Table 2
Restrictions on possible values of X for $N > 1$.

	Restrictions on N, n_X	Number of possible X
$\omega_Y^2 \neq \omega_Z^2$	$N \leq 10$, n_X odd and even	31
$\omega_Y = \omega_Z \neq 0$	$N \leq 10$, n_X odd and even, $N = 11, 15, 21$, n_X odd	46
$\omega_Y = \omega_Z = 0$ with $\omega_X \neq 0$ or $\omega_4 \neq 0$	$N \leq 15$, n_X odd and even	71

Table 3
Admissible values of good coordinates.

	good X	good Y	good Z
(A)	$\mathcal{S}_1$	$\mathcal{S}_1$	$\mathcal{S}_1$
(B)	$\mathcal{S}_2$	$\mathcal{S}_1$	$\mathcal{S}_1$
(C)	$\mathcal{S}_3$	$\mathcal{S}_1$	$\mathcal{S}_1$
(D)	$\mathcal{S}_2$	$\mathcal{S}_2$	$\mathcal{S}_2$
(E)	$\mathcal{S}_3$	$\mathcal{S}_3$	$\mathcal{S}_3$

Let us summarize the results of this subsection. Given a finite orbit O, common coordinate X of all points of any 2-colored suborbit $O_{yz} \subset O$ of length $N > 1$ has the form $X = 2\cos \pi n_X/N$, $0 < n_X < N$, where N and n_X are coprime. Unless $\omega_X = \omega_Y = \omega_Z = \omega_4 = 0$, one has a number of restrictions on possible values of N and n_X listed in Table 2. These restrictions imply in particular that X can take only a finite number of explicitly defined values. In the next subsection, we use this observation to construct an exhaustive search algorithm giving all finite orbits of (14).

2.6. Search algorithm

Let $O \subset \mathbb{C}^3$ be a finite orbit of the induced $\bar{\Lambda}$ action (14) consisting of more than one point. Since we are interested in nonequivalent orbits, it can be assumed that the parameters $\omega_{X,Y,Z,4}$ satisfy one of the following sets of constraints:

(A) $\omega_X^2 \neq \omega_Y^2 \neq \omega_Z^2$,
(B) $\omega_X^2 \neq \omega_Y^2$, $\omega_Y = \omega_Z \neq 0$,
(C) $\omega_X \neq 0$, $\omega_Y = \omega_Z = 0$,
(D) $\omega_X = \omega_Y = \omega_Z \neq 0$,
(E) $\omega_X = \omega_Y = \omega_Z = 0$, $\omega_4 \neq 0$,
(F) $\omega_X = \omega_Y = \omega_Z = \omega_4 = 0$.

In what follows, the case (F) will be omitted, as all finite orbits with such parameter values have already been described above.

Definition 35. Let $\mathbf{r} = (X, Y, Z)$ be a point in O. Its coordinate X (or Y, Z) will be called *good* if $\mathbf{r}$ is not fixed by at least one of the transformations y and z (resp. x and z, x and y).

Remark 36. All coordinates of a good point are good. If $\mathbf{r}$ is a bad point, e.g. fixed by y and z but not by x, then it has good coordinates Y and Z.

Define three finite sets of numbers (cf. Table 2):

$$\mathcal{S}_1 = \left\{ 2\cos\frac{\pi n}{N} \Big| 1 < N \leq 10, n \text{ odd and even} \right\},$$
$$\mathcal{S}_2 = \left\{ 2\cos\frac{\pi n}{N} \Big| 1 < N \leq 10, n \text{ odd and even}; N = 11, 15, 21, n \text{ odd} \right\},$$
$$\mathcal{S}_3 = \left\{ 2\cos\frac{\pi n}{N} \Big| 1 < N \leq 15, n \text{ odd and even} \right\}.$$

In all three cases n is supposed to be coprime with N and $0 < n < N$. Now the results of the previous subsection imply that good coordinates of any point $\mathbf{r} \in O$ belong to one of these lists according to Table 3.

Any orbit O is completely defined by a point $\mathbf{r} \in O$ and the parameter triple $\boldsymbol{\omega} = (\omega_X, \omega_Y, \omega_Z)$. Equivalently, instead of $\boldsymbol{\omega}$ one can use three points $x(\mathbf{r}), y(\mathbf{r}), z(\mathbf{r})$ (some of them can coincide with $\mathbf{r}$). Denote

$$X' = X(x(\mathbf{r})), \qquad Y' = Y(y(\mathbf{r})), \qquad Z' = Z(z(\mathbf{r})), \tag{64}$$

then we have

$$\omega_X = X + X' + YZ, \qquad \omega_Y = Y + Y' + XZ, \qquad \omega_Z = Z + Z' + XY. \tag{65}$$

Definition 37. Let $\mathbf{r}$ be a good point in a finite orbit O. The set of four points $\{\mathbf{r}, x(\mathbf{r}), y(\mathbf{r}), z(\mathbf{r})\}$ will be called a good generating configuration (GGC) for O if at least two of three points $x(\mathbf{r}), y(\mathbf{r}), z(\mathbf{r})$ are good.

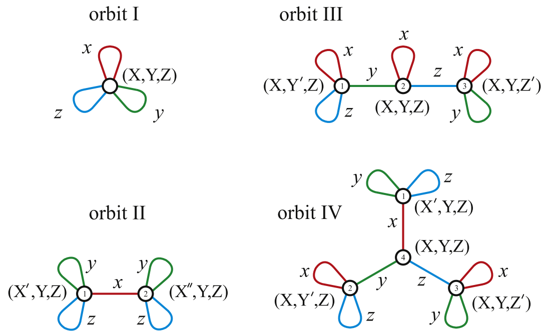


Fig. 7. Four orbits without GGCs.

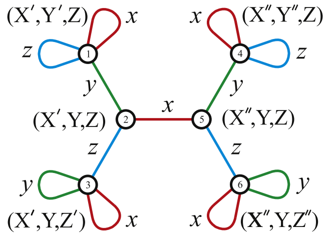


Fig. 8. 6-vertex graph without GGCs.

Lemma 38. *Let O be a finite orbit that does not contain a GGC. Then $\Sigma(O)$ can only be equivalent (up to permutations of colors) to one of the four graphs shown in Fig. 7.*

Proof. If O contains more than 2 points, then at least one of them is good. Denoting this point by $\mathbf{r}$, we can assume that $y(\mathbf{r})$ and $z(\mathbf{r})$ are bad. Now if $x(\mathbf{r}) = \mathbf{r}$, then one obtains orbit III. The case when $x(\mathbf{r}) \neq \mathbf{r}$ is bad corresponds to orbit IV. Finally, if $x(\mathbf{r}) \neq \mathbf{r}$ is another good point, then by assumptions of the Lemma the points $y(x(\mathbf{r}))$ and $z(x(\mathbf{r}))$ are bad, and $\Sigma(O)$ is given by the 6-vertex graph represented in Fig. 8.

It turns out, however, that this last graph is forbidden. To see this, note that yz-suborbits 1-2-3 and 4-5-6 both have length 3, therefore X' and X'' are equal to ± 1. Since $X' \neq X''$, one can set $X' = 1, X'' = -1$. Then from the relations corresponding to y- and z-edges,

$$\omega_Y = Y + Y' + X'Z = 2Y + X'Z' = 2Y + X''Z'' = Y + Y'' + X''Z,$$
$$\omega_Z = Z + Z' + X'Y = 2Z + X'Y' = 2Z + X''Y'' = Z + Z'' + X''Y,$$

it follows that $Y = -Z' = Z''$ and $Z = -Y' = Y''$. Self-loops of color x at the points 1, 3, 4 and 6 in turn imply that $\omega_X = 0$, $Y^2 = Z^2 = 2$. However, this is incompatible with the x-edge 2-5, which gives $\omega_X = YZ$. □

The orbits of (14) with graphs I–IV are completely described by the following:

Lemma 39. 1. *Orbits of type I consist of one point $(X, Y, Z) \in \mathbb{C}^3$. The parameters $\omega_{X,Y,Z,4}$ are given by*

$$\omega_X = 2X + YZ, \qquad \omega_Y = 2Y + XZ, \qquad \omega_Z = 2Z + XY, \tag{66}$$

$$\omega_4 = 4 + 2XYZ + X^2 + Y^2 + Z^2. \tag{67}$$

2. *Any orbit of type* II *is equivalent to an orbit consisting of* 2 *points* $(X', 0, 0)$ *and* $(X'', 0, 0)$, *where* $X', X'' \in \mathbb{C}$, $X' \neq X''$ *and* $\omega_X = X' + X''$, $\omega_Y = \omega_Z = 0$, $\omega_4 = 4 + X'X''$.

3. *Any orbit of type* III *is equivalent to an orbit consisting of* 3 *points* $(1, 0, 0)$, $(1, \omega, 0)$, $(1, 0, \omega)$, *where* $\omega \in \mathbb{C}^*$ *and* $\omega_X = 2$, $\omega_Y = \omega_Z = \omega$, $\omega_4 = 5$.

4. *Any orbit of type* IV *is equivalent to an orbit consisting of* 4 *points* $(1, 1, 1)$, $(\omega - 2, 1, 1)$, $(1, \omega - 2, 1)$, $(1, 1, \omega - 2)$, *where* $\omega \in \mathbb{C}$, $\omega \neq 3$ *and* $\omega_X = \omega_Y = \omega_Z = \omega$, $\omega_4 = 3\omega$.

Proof. Statement 1 is obvious (ω_4 is determined from (10)), hence we start with orbits of type II. In this case, since xy- and xz-suborbits 1-2 have length 2, one finds $Y = Z = 0$. From the relations corresponding to the self-loops then follows $\omega_Y = \omega_Z = 0$.

For orbits of type III, xy-suborbit 1-2 and xz-suborbit 2-3 both have length 2, therefore $Y = Z = 0$. Similarly, yz-suborbit 1-2-3 has length 3 and thus $X = \pm 1$. Since the simultaneous change of signs of e.g. ω_X, ω_Y, and also X- and Y-coordinates of all points leads to an equivalent orbit, one can set $X = 1$, and then x-self-loop at the point 2 gives $\omega_X = 2$. At last, y- and z-edges of the graph imply that $\omega_Y = \omega_Z = Y' = Z'$.

In graph IV, xy-suborbit 1-4-2, xz-suborbit 1-4-3 and yz-suborbit 2-4-3 have length 3, therefore X, Y and Z are equal to ± 1. It can be assumed that either (a) $X = Y = Z = 1$ or (b) $X = Y = Z = -1$. In the case (a), y- and z-self-loops at the point 1 imply that $\omega_Y = \omega_Z = 2 + X'$, hence by symmetry

$$\omega_X = \omega_Y = \omega_Z = 2 + X' = 2 + Y' = 2 + Z',$$

and the relations corresponding to the edges 1-4, 2-4 and 3-4 are satisfied automatically. In the case (b), one similarly finds $\omega_X = \omega_Y = \omega_Z = -2 - X' = -2 - Y' = -2 - Z'$, but e.g. the relation 1-2 gives $\omega_X = X'$. Thus $X' = -1$ and we obtain a contradiction. $\square$

Unless $\omega_X = \omega_Y = \omega_Z = \omega_4 = 0$, one has only a finite number of GGCs (and hence only a finite number of finite orbits different from I to IV). Indeed, these configurations can be of two types:

Type (i). All four points $\mathbf{r}, x(\mathbf{r}), y(\mathbf{r}), z(\mathbf{r}) \in O$ are good. In this case six coordinates X, Y, Z, X', Y', Z' (defined by (64)) are good, hence each of them can take only a finite number of values, as specified in Table 3.

Type (ii). One of three points $x(\mathbf{r}), y(\mathbf{r}), z(\mathbf{r}) \in O$ is bad. Suppose e.g. that $x(\mathbf{r})$ is bad, then X, Y, Z, Y', Z' are good coordinates, but X' is not. However, since $x(\mathbf{r})$ is fixed by y and z, we have the equations

$$\begin{cases} \omega_Y = 2Y + X'Z = Y + Y' + XZ, \\ \omega_Z = 2Z + X'Y = Z + Z' + XY. \end{cases} \tag{68}$$

Unless $Y = Z = 0$, one can use (68) to express X' in terms of good coordinates. Also notice that Y, Z, Y', Z' should satisfy an additional relation

$$Y(Y - Y') = Z(Z - Z'). \tag{69}$$

On the other hand if $Y = Z = 0$, then (68) implies that $\omega_Y = \omega_Z = 0$, the orbit O is of type II and in particular it does not contain a GGC.

Let us now describe in more detail the sets of good coordinates generating all possible candidates for finite orbits, different from orbits I–IV and those of Cayley type:

Class 1 (A-i). Here one has $31^6 \approx 10^9$ GGCs, corresponding to all possible $X, Y, Z, X', Y', Z' \in \mathcal{S}_1$. Since we are interested in nonequivalent orbits, it can be assumed that either $0 \le X \le Y \le Z$ or $0 \ge X \ge Y \ge Z$, and then the above number reduces to $16 \cdot 17 \cdot 18 \cdot 31^3/3 - 1 = 48618911$. We do not exclude the remaining equivalent GGCs for simplicity of the algorithm.

Class 2 (A-ii, B-ii, C-ii, D-ii, E-ii). In this case it is convenient to deal not only with $\omega_{X,Y,Z}$ satisfying one of the conditions (A)–(E), but also with equivalent parameter triples. One can then assume that $x(\mathbf{r})$ is bad and $0 \le Y \le Z, Z > 0$. Since Z' can now be determined from (69), the whole orbit is completely fixed by four good coordinates X, Y, Z, Y', taking their values in the set

$$\mathcal{S}_4 = \left\{ 2\cos\frac{\pi n}{N} \Big| 1 < N \le 15, N = 21, n \text{ odd and even} \right\},$$

consisting of 83 elements. The total number of configurations to be checked is therefore equal to $41 \cdot 22 \cdot 83^2 = 6213878$.

Class 3 (B-i, C-i). Here we use good coordinates $X, X' \in \mathcal{S}_4$, $Y, Y', Z \in \mathcal{S}_1$, while Z' is computed from

$$Z' = \left(Y + Y' + XZ\right) - Z - XY,$$

and it can be assumed that $0 \le |Y| \le Z$. This gives $16^2 \cdot 31 \cdot 83^2 = 54671104$ configurations, from which we should choose only those with $Z' \in \mathcal{S}_1$.

Class 4 (D-i, E-i). These orbits are completely determined by $X, X', Y, Z \in \mathcal{S}_4$. Since $x(\mathbf{r}), y(\mathbf{r}), z(\mathbf{r})$ are good, it can be assumed that $X \le Y \le Z$, which leads to $83^2 \cdot 84 \cdot 85/6 = 8197910$ possibilities.

In order to check which generating sets do actually lead to finite orbits, one can use the following algorithm:

1. Consider any generating set from the above as a set $\mathcal{P}$ of known orbit *points* and known *adjacency relations* between them. E.g. if it is known by construction that $x(\mathbf{r}) = \mathbf{r}'$ for some $\mathbf{r}, \mathbf{r}' \in \mathcal{P}$, we will say that $\mathbf{r}'$ is a known x-neighbor of $\mathbf{r}$ and vice versa. Thus any point $\mathbf{r} \in \mathcal{P}$ has at most 3 known neighbors, corresponding to x-, y- and z-edges originating from $\mathbf{r}$.
2. If the set is characterized by $\omega_X = \omega_Y = \omega_Z = \omega_4 = 0$, the algorithm stops (the only finite orbits with such parameters are Cayley orbits).
3. Using $\mathcal{P}$, construct the set $\mathcal{P}^u$ of points with at least one unknown neighbor.
4. Choose an arbitrary point $\mathbf{r} = (X, Y, Z) \in \mathcal{P}^u$. Assume for definiteness that its x-neighbor $x(\mathbf{r}) = (X', Y, Z)$ is unknown. Then compute X' and proceed as follows:
   4.1. If $(X', Y, Z) \in \mathcal{P}^u$, then add the appropriate x-adjacency relation to $\mathcal{P}$, update $\mathcal{P}^u$ and go to Step 5, else
   4.2. If X' has a good value (in practice it is sufficient to require $X' \in \mathcal{S}_4$), then add (X', Y, Z) and the appropriate x-adjacency relation to $\mathcal{P}$, update $\mathcal{P}^u$ and go to Step 4, else

4.3. If $2Y + X'Z = \omega_Y$ and $2Z + X'Y = \omega_Z$, then add (X', Y, Z) and the appropriate x-, y- and z-adjacency relations to $\mathscr{P}$, update $\mathscr{P}^u$ and go to Step 5, else the algorithm stops (the orbit cannot be finite).

5. If $\mathscr{P}^u$ is empty, the algorithm stops (the orbit is finite and its points are given by $\mathscr{P}$), otherwise go to Step 4.

Remark 40. It is easy to see that the algorithm stops after a finite number of steps. Indeed, the total number N_g of good points in any finite orbit which is not of Cayley type cannot exceed $71^2 \cdot 2 = 10082$, while the number of bad points cannot exceed $N_g + 2$.

2.7. List of finite orbits

We have implemented the above algorithm with a computer program written in C language. The check of all generating sets took less than 10 min on a usual 1.7 GHz desktop computer. It turned out that there are only 45 nonequivalent finite exceptional orbits, different from orbits I–IV and Cayley orbits. We describe these orbits in Table 4 by indicating one of the orbit points

$$(X, Y, Z) = \big(2\cos\pi r_X, 2\cos\pi r_Y, 2\cos\pi r_Z\big),$$

and the parameter triple $(\omega_X, \omega_Y, \omega_Z)$. For further convenience, we also include the value of $4 - \omega_4$, computed from the Jimbo–Fricke relation (10). The graphs of exceptional $\bar{\Lambda}$ orbits are shown in Figs. 9–11 (marked vertices correspond to the points listed in Table 4).

Our results can now be summarized in

Theorem 1. *The list of all nonequivalent finite orbits of the induced $\bar{\Lambda}$ action (14) consists of the following:*

- *four orbits* I–IV *, described in Lemma 39;*
- *45 exceptional orbits listed in Table 4;*
- *Cayley orbits; all of these can be generated from the points*

$$\big(-2\cos\pi(r_Y + r_Z), 2\cos\pi r_Y, 2\cos\pi r_Z\big), \quad r_{Y,Z} \in \mathbb{Q}$$

with $\omega_X = \omega_Y = \omega_Z = 0$ (the relation $\omega_4 = 0$ is satisfied automatically).

Remark 41. Note that the graphs of orbits I–IV and of all exceptional orbits except orbits 30, 43–45 contain self-loops. It means in particular that these orbits do not split under the action of non-extended modular group Λ. In fact the last statement holds for orbits 30, 43–45 as well, because in all four cases the orbit graphs contain simple cycles with an odd number of edges.

We now turn to the description of nonequivalent finite orbits of the $\bar{\Lambda}$ action (7) on $\mathcal{M}$. Note that, given $\omega_{X,Y,Z,4}$, Eqs. (11)–(12) have only a finite number of solutions for $\{p_x, p_y, p_z, p_\infty\}$. In fact this number cannot exceed 24, see proof of Proposition 10, and all such solutions are related by the affine D_4 transformations. A natural question is therefore: when does the 7-tuple $\mathbf{p} = (p_x, p_y, p_z, p_\infty, X, Y, Z)$ (see (8), (9)) completely fix the conjugacy class of the triple $(M_x, M_y, M_z) \in G^3$, $G = SL(2, \mathbb{C})$ in $\mathcal{M} = G^3/G$?

Let us first prove an auxiliary result:

Lemma 42. *Let $M^a, M^b, M^c \in G$ be three matrices such that the eigenvalues of at least one of them are different from ± 1. Then one and only one of the following holds:*

1. *Seven quantities*

$$t_a = \operatorname{Tr} M^a, \qquad t_b = \operatorname{Tr} M^b, \qquad t_c = \operatorname{Tr} M^c, \qquad t_{abc} = \operatorname{Tr}\left(M^a M^b M^c\right), \tag{70}$$

$$t_{ab} = \operatorname{Tr}\left(M^a M^b\right), \qquad t_{ac} = \operatorname{Tr}\left(M^a M^c\right), \qquad t_{bc} = \operatorname{Tr}\left(M^b M^c\right), \tag{71}$$

completely fix the conjugacy class of the triple (M^a, M^b, M^c) in $\mathcal{M}$;

2. *M^a, M^b, M^c have a common eigenvector.*

Proof. Using the same tricks as in the proof of Lemma 5, one easily expresses $t_{bac} = \operatorname{Tr}\left(M^b M^a M^c\right)$ in terms of (70)–(71):

$$\begin{aligned} t_{bac} &= \operatorname{Tr}\left(\left[t_{ab}\mathbf{1} - (M^a)^{-1}(M^b)^{-1}\right] M^c\right) = t_{ab}t_c - \operatorname{Tr}\left((t_a\mathbf{1} - M^a)(t_b\mathbf{1} - M^b)M^c\right) \\ &= t_{ab}t_c + t_{ac}t_b + t_{bc}t_a - t_a t_b\, t_c - t_{abc}. \end{aligned}$$

We may therefore assume without loss of generality that the eigenvalues of M^a are not equal to ± 1; in particular, M^a is diagonalizable. It is convenient to transform it into diagonal form $M^a = \operatorname{diag}(\lambda_a, \lambda_a^{-1})$, with λ_a fixed by t_a. Now the equations for t_b and t_{ab} (t_c and t_{ac}) fix M^b_{11}, M^b_{22} and $M^b_{12}M^b_{21}$ (resp. M^c_{11}, M^c_{22} and $M^c_{12}M^c_{21}$). The equations for t_{bc} and t_{abc}, in their turn, completely determine $(M^bM^c)_{11}$ and $(M^bM^c)_{22}$, hence the products $M^b_{12}M^c_{21}$ and $M^b_{21}M^c_{12}$ are also fixed.

Table 4
Exceptional finite $\bar{\Lambda}$ orbits.

	Size	$(\omega_X, \omega_Y, \omega_Z, 4-\omega_4)$	(r_X, r_Y, r_Z)
1	5	$(0, 1, 1, 0)$	$(2/3, 1/3, 1/3)$
2	5	$(3, 2, 2, -3)$	$(1/3, 1/3, 1/3)$
3	6	$(1, 0, 0, 2)$	$(1/2, 1/3, 1/3)$
4	6	$(\sqrt{2}, 0, 0, 1)$	$(1/4, 1/3, 3/4)$
5	6	$(3, 2\sqrt{2}, 2\sqrt{2}, -4)$	$(1/2, 1/4, 1/4)$
6	6	$\left(1-\sqrt{5}, \frac{3-\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}, -2+\sqrt{5}\right)$	$(4/5, 1/3, 1/3)$
7	6	$\left(1+\sqrt{5}, \frac{3+\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}, -2-\sqrt{5}\right)$	$(2/5, 1/3, 1/3)$
8	7	$(1, 1, 1, 0)$	$(1/2, 1/2, 1/2)$
9	8	$(2, 0, 0, 0)$	$(0, 1/3, 2/3)$
10	8	$(1, \sqrt{2}, \sqrt{2}, 0)$	$(1/2, 1/2, 1/2)$
11	8	$\left(\frac{3+\sqrt{5}}{2}, 1, 1, -\frac{\sqrt{5}+1}{2}\right)$	$(1/3, 1/2, 1/2)$
12	8	$\left(\frac{3-\sqrt{5}}{2}, 1, 1, \frac{\sqrt{5}-1}{2}\right)$	$(1/3, 1/2, 1/2)$
13	9	$\left(2-\sqrt{5}, 2-\sqrt{5}, 2-\sqrt{5}, \frac{5\sqrt{5}-7}{2}\right)$	$(4/5, 3/5, 3/5)$
14	9	$\left(2+\sqrt{5}, 2+\sqrt{5}, 2+\sqrt{5}, -\frac{5\sqrt{5}+7}{2}\right)$	$(2/5, 1/5, 1/5)$
15	10	$(1, 0, 0, 1)$	$(1/3, 1/3, 2/3)$
16	10	$\left(3-\sqrt{5}, 3-\sqrt{5}, 3-\sqrt{5}, \frac{7\sqrt{5}-11}{2}\right)$	$(3/5, 3/5, 3/5)$
17	10	$\left(3+\sqrt{5}, 3+\sqrt{5}, 3+\sqrt{5}, -\frac{7\sqrt{5}+11}{2}\right)$	$(1/5, 1/5, 1/5)$
18	10	$\left(-\frac{\sqrt{5}-1}{2}, -\frac{\sqrt{5}-1}{2}, -\frac{\sqrt{5}-1}{2}, 0\right)$	$(1/2, 1/2, 1/2)$
19	10	$\left(\frac{\sqrt{5}+1}{2}, \frac{\sqrt{5}+1}{2}, \frac{\sqrt{5}+1}{2}, 0\right)$	$(1/2, 1/2, 1/2)$
20	12	$(0, 0, 0, 3)$	$(2/3, 1/4, 1/4)$
21	12	$(1, 0, 0, 2)$	$(0, 1/4, 3/4)$
22	12	$(2, \sqrt{5}, \sqrt{5}, -2)$	$(1/5, 2/5, 2/5)$
23	12	$\left(\frac{3+\sqrt{5}}{2}, \frac{\sqrt{5}+1}{2}, \frac{\sqrt{5}+1}{2}, -\sqrt{5}\right)$	$(2/5, 2/5, 2/5)$
24	12	$\left(\frac{3-\sqrt{5}}{2}, -\frac{\sqrt{5}-1}{2}, -\frac{\sqrt{5}-1}{2}, \sqrt{5}\right)$	$(4/5, 4/5, 4/5)$
25	12	$\left(\frac{\sqrt{5}+1}{2}, \frac{\sqrt{5}-1}{2}, 1, 0\right)$	$(1/2, 1/2, 1/2)$
26	15	$\left(\frac{3-\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}, \sqrt{5}-1\right)$	$(1/2, 3/5, 3/5)$
27	15	$\left(\frac{3+\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}, -\sqrt{5}-1\right)$	$(1/2, 1/5, 1/5)$
28	15	$\left(\frac{5-\sqrt{5}}{2}, 1-\sqrt{5}, 1-\sqrt{5}, \frac{3\sqrt{5}-5}{2}\right)$	$(3/5, 4/5, 4/5)$
29	15	$\left(\frac{5+\sqrt{5}}{2}, 1+\sqrt{5}, 1+\sqrt{5}, -\frac{3\sqrt{5}+5}{2}\right)$	$(1/5, 2/5, 2/5)$
30	16	$(0, 0, 0, 2)$	$(2/3, 2/3, 2/3)$
31	18	$(2, 2, 2, -1)$	$(0, 1/5, 3/5)$
32	18	$(1-2\cos 2\pi/7, 1-2\cos 2\pi/7, 1-2\cos 2\pi/7, 4\cos 2\pi/7)$	$(6/7, 5/7, 5/7)$
33	18	$(1-2\cos 4\pi/7, 1-2\cos 4\pi/7, 1-2\cos 4\pi/7, 4\cos 4\pi/7)$	$(2/7, 3/7, 3/7)$
34	18	$(1-2\cos 6\pi/7, 1-2\cos 6\pi/7, 1-2\cos 6\pi/7, 4\cos 6\pi/7)$	$(4/7, 1/7, 1/7)$
35	20	$\left(\frac{3-\sqrt{5}}{2}, 0, 0, 1+\sqrt{5}\right)$	$(0, 1/3, 2/3)$
36	20	$\left(\frac{3+\sqrt{5}}{2}, 0, 0, 1-\sqrt{5}\right)$	$(0, 1/3, 2/3)$
37	20	$\left(1, -\frac{\sqrt{5}-1}{2}, -\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}+1}{2}\right)$	$(2/3, 3/5, 3/5)$
38	20	$\left(1, \frac{\sqrt{5}+1}{2}, \frac{\sqrt{5}+1}{2}, -\frac{\sqrt{5}-1}{2}\right)$	$(2/3, 1/5, 1/5)$

(continued on next page)

Table 4 (*continued*)

	Size	$(\omega_X, \omega_Y, \omega_Z, 4-\omega_4)$	(r_X, r_Y, r_Z)
39	24	$(1, 1, 1, 1)$	$(1/5, 1/2, 1/2)$
40	30	$\left(-\frac{\sqrt{5}+1}{2}, 0, 0, \frac{3-\sqrt{5}}{2}\right)$	$(2/3, 2/3, 2/3)$
41	30	$\left(\frac{\sqrt{5}-1}{2}, 0, 0, \frac{3+\sqrt{5}}{2}\right)$	$(2/3, 2/3, 2/3)$
42	36	$(1, 0, 0, 2)$	$(0, 1/5, 4/5)$
43	40	$\left(0, 0, 0, \frac{5-\sqrt{5}}{2}\right)$	$(2/5, 2/5, 2/5)$
44	40	$\left(0, 0, 0, \frac{5+\sqrt{5}}{2}\right)$	$(4/5, 4/5, 4/5)$
45	72	$(0, 0, 0, 3)$	$(1/2, 1/5, 2/5)$

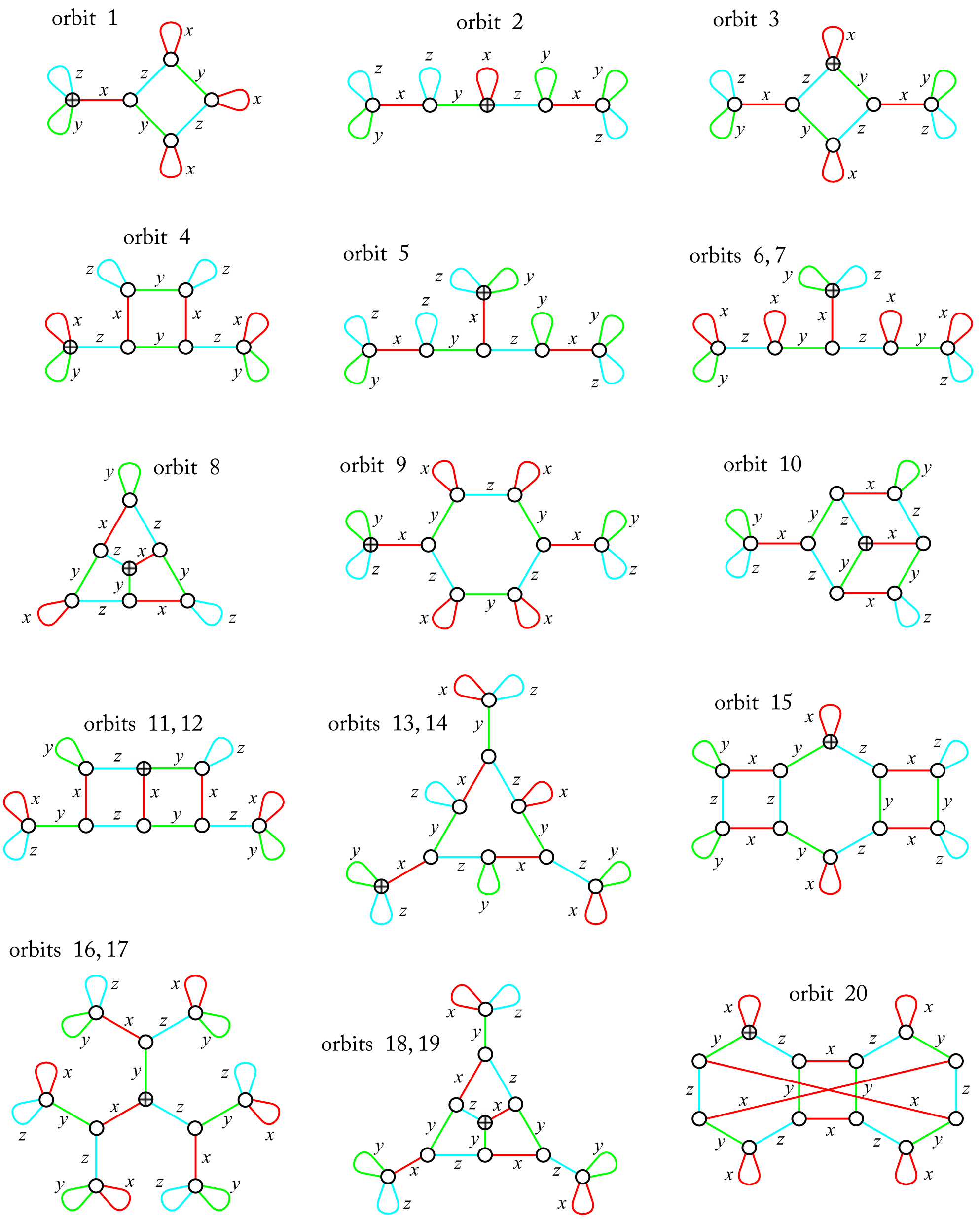


Fig. 9. Graphs of exceptional orbits 1–20.

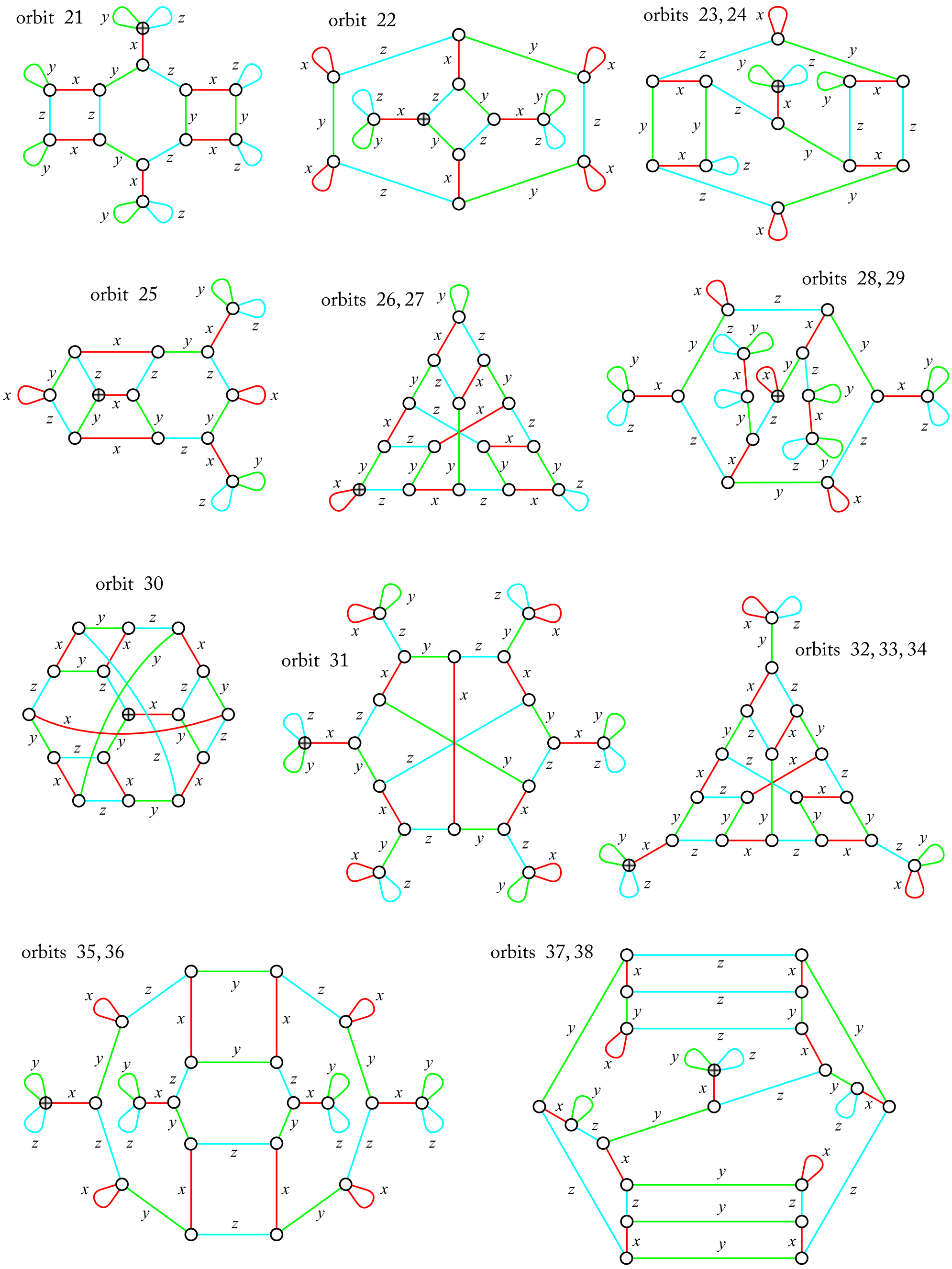


Fig. 10. Graphs of exceptional orbits 21–38.

If $M^b_{12}M^b_{21} = M^c_{12}M^c_{21} = M^b_{12}M^c_{21} = M^b_{21}M^c_{12} = 0$, then either $M^b_{12} = M^c_{12} = 0$ or $M^b_{21} = M^c_{21} = 0$, i.e. $M^{a,b,c}$ are simultaneously lower or upper triangular. On the other hand if at least one of the four products, say $M^b_{12}M^b_{21}$, is non-zero, then, using the remaining freedom of conjugation of $M^{a,b,c}$ by any diagonal matrix, one can set $M^b_{12} = 1$ and $M^b_{21} (\neq 0)$, M^c_{12} and M^c_{21} become completely fixed. Moreover, in this case $M^{a,b,c}$ clearly cannot have a common eigenvector. $\square$

Lemma 43. *Let $M_x, M_y, M_z \in G$. One and only one of the following holds:*

1. *Conjugacy class of the triple (M_x, M_y, M_z) in $\mathcal{M}$ is uniquely fixed by the 7-tuple $\mathbf{p} = (p_x, p_y, p_z, p_\infty, X, Y, Z)$, defined by (8)–(9).*
2. *M_x, M_y, M_z have a common eigenvector.*

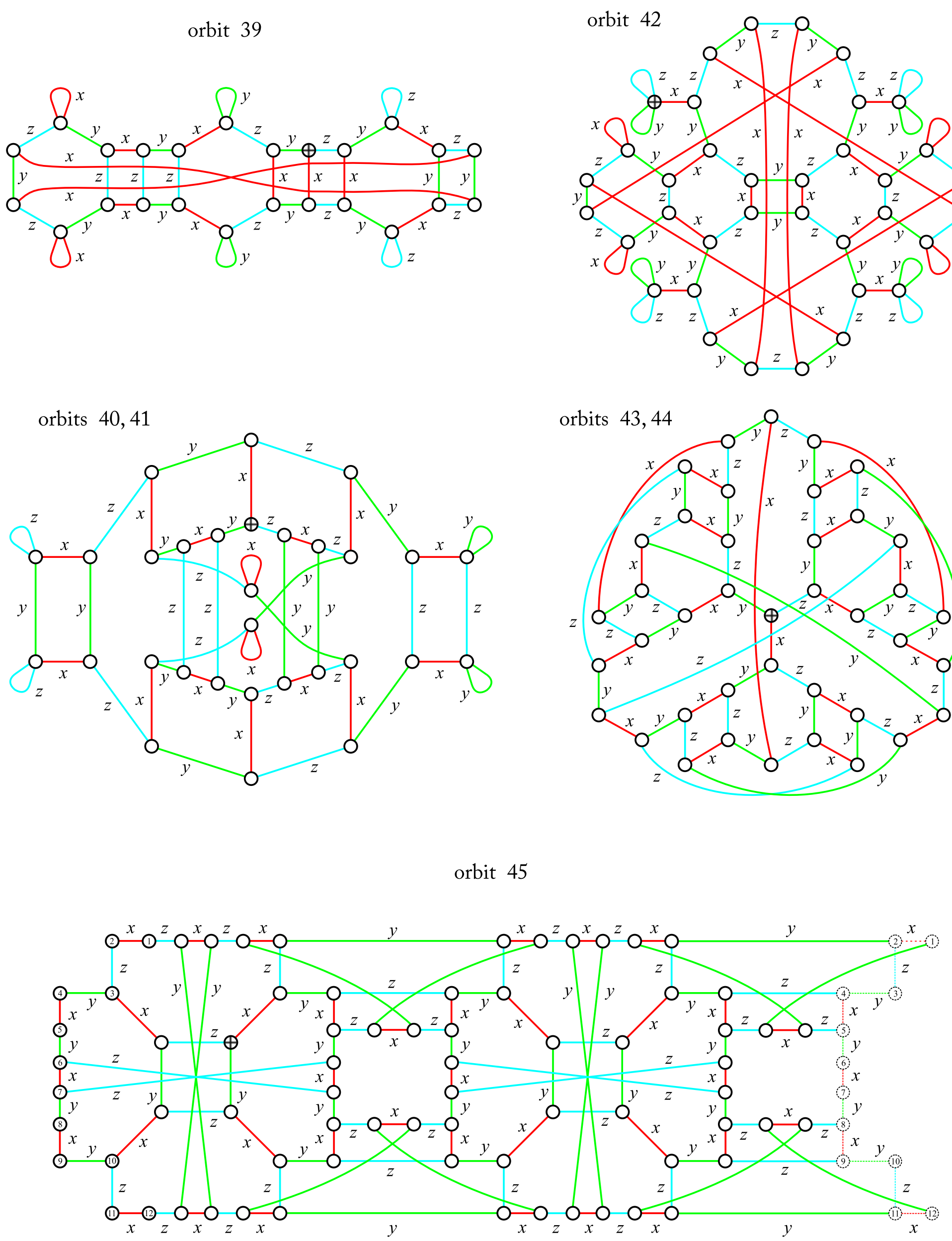


Fig. 11. Graphs of exceptional orbits 39–45.

Proof. When the eigenvalues of at least one of three matrices $M_{x,y,z}$ are not equal to ±1, the statement is equivalent to the previous lemma.

Similarly, if e.g. the eigenvalues of M_xM_y (or $M_xM_y^{-1}$) are different from ±1, we can apply Lemma 42 to the triple $M^a = M_xM_y$, $M^b = M_y^{-1}$, $M^c = M_z$ (resp. $M^a = M_xM_y^{-1}$, $M^b = M_y$, $M^c = M_z$). Since $t_a, t_b, t_c, t_{ab}, t_{ac}, t_{bc}, t_{abc}$ are clearly expressible in terms of $\mathbf{p}$, the conjugacy class of (M^a, M^b, M^c), and hence that of (M_x, M_y, M_z), is fixed unless $M^{a,b,c}$ can be simultaneously brought to lower or upper triangular form.

Therefore, it is sufficient to prove the lemma in the case when the eigenvalues of $M_{x,y,z}$, $M_xM_y^{\pm1}$, $M_xM_z^{\pm1}$ and $M_yM_z^{\pm1}$ are equal to ±1. We can assume without loss of generality that $\operatorname{Tr} M_x = \operatorname{Tr} M_y = \operatorname{Tr} M_z = 2$, but then from the relation $\operatorname{Tr}\left(M_xM_y\right) + \operatorname{Tr}\left(M_xM_y^{-1}\right) = \operatorname{Tr} M_x \cdot \operatorname{Tr} M_y$ follows that $\operatorname{Tr}\left(M_xM_y\right) = 2$. Similarly, one has $\operatorname{Tr}\left(M_xM_z\right) = \operatorname{Tr}\left(M_yM_z\right) = 2$. Now, if we transform M_x into upper triangular form, the relations $\operatorname{Tr} M_x = \operatorname{Tr} M_y = \operatorname{Tr}\left(M_xM_y\right) = 2$ imply that either M_x is the identity matrix or M_y is also upper triangular. Combining with analogous result for M_x, M_z we see that all three matrices should have a common eigenvector. $\square$

Lemma 44. *If three matrices $M_x, M_y, M_z \in G$ have a common eigenvector, then the elements of* $\mathbf{p}$ *satisfy characteristic relations (66) of orbit* I*, with $\omega_{X,Y,Z}$ defined by (11).*

Proof. Transforming $M_{x,y,z}$ into upper triangular form, we see that $\mathbf{p}$ can be written in terms of the eigenvalues of $M_{x,y,z}$. It is sufficient to substitute these expressions into the relations (66) to check that they are satisfied automatically. □

We now formulate a converse statement:

Lemma 45. *Let $M_x, M_y, M_z \in G$ be three matrices with no common eigenvector. If* $\mathbf{p}$ *satisfies the relations (66), then at least one of four matrices M_x, M_y, M_z, $M_zM_yM_x$ is equal to $\pm\mathbf{1}$.*

Proof. Using (66) and (10), write ω_4 in terms of X, Y, Z:

$$\omega_4 = 4 + 2XYZ + X^2 + Y^2 + Z^2.$$

Substituting the expressions for $\omega_{X,Y,Z,4}$ into the cubic equation (15) for $\xi = p_x^2 + p_y^2 + p_z^2 + p_\infty^2$, one finds that it has only two solutions: (1) $\xi = 8 + XYZ$ and (2) $\xi = 4 + X^2 + Y^2 + Z^2$.

Case(1). Let us write $X = 2\cos\pi r_X$, $Y = 2\cos\pi r_Y$, $Z = 2\cos\pi r_Z$. It is straightforward to check that $(p_x^0, p_y^0, p_z^0, p_\infty^0)$ defined by

$$p_x^0 = 2\cos\pi(r_Y + r_Z - r_X)/2, \quad p_y^0 = 2\cos\pi(r_X + r_Z - r_Y)/2,$$
$$p_z^0 = 2\cos\pi(r_X + r_Y - r_Z)/2, \quad p_\infty^0 = 2\cos\pi(r_X + r_Z + r_Y)/2,$$

is one of possible solutions for $(p_x, p_y, p_z, p_\infty)$. All other solutions characterized by the same value of ξ have the form (17), see the proof of Proposition 10. However, it is not difficult to show that for all such $(p_x, p_y, p_z, p_\infty)$ one can find infinitely many triples (M_x', M_y', M_z') of upper triangular matrices with the same $\mathbf{p}$ as (M_x, M_y, M_z). E.g. if $p_\nu = p_\nu^0$, $\nu = x, y, z, \infty$, then we may set

$$M_x' = \begin{pmatrix} e^{i\pi(r_Y+r_Z-r_X)/2} & * \\ 0 & e^{-i\pi(r_Y+r_Z-r_X)/2} \end{pmatrix},$$
$$M_y' = \begin{pmatrix} e^{i\pi(r_X+r_Z-r_Y)/2} & * \\ 0 & e^{-i\pi(r_X+r_Z-r_Y)/2} \end{pmatrix},$$
$$M_z' = \begin{pmatrix} e^{i\pi(r_X+r_Y-r_Z)/2} & * \\ 0 & e^{-i\pi(r_X+r_Y-r_Z)/2} \end{pmatrix}.$$

Now since $\mathbf{p}$ does not fix the conjugacy class of the triple (M_x, M_y, M_z) uniquely, by Lemma 43 $M_{x,y,z}$ should have a common eigenvector.

Case(2). Here, one possible solution for $(p_x, p_y, p_z, p_\infty)$ is

$$p_x^0 = X, \qquad p_y^0 = Y, \qquad p_z^0 = Z, \qquad p_\infty^0 = 2, \tag{72}$$

and all the others are given by (17). Consider the solution (72) and transform $M_zM_yM_x$ into upper triangular form: $M_zM_yM_x = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$. Since

$$X = \operatorname{Tr}\left(M_yM_z\right) = \operatorname{Tr}\left(M_zM_yM_x \cdot M_x^{-1}\right) = p_x - \alpha\,(M_x)_{21},$$

the relation $p_x = X$ implies that either $M_zM_yM_x = \mathbf{1}$ or M_x is upper triangular. Repeating the same procedure with $p_y = Y$, $p_z = Z$ and using the assumption that $M_{x,y,z}$ have no common eigenvectors, one concludes that $M_zM_yM_x = \mathbf{1}$. Other solutions for $(p_x, p_y, p_z, p_\infty)$ are treated in a similar manner. □

We thus obtain a description of all nonequivalent finite orbits of the $\bar{\Lambda}$ action (7) on $\mathcal{M}$:

- There are two families of nonequivalent orbits that consist of one point. They are given by the conjugacy classes of triples (a) $(\mathbf{1}, M_y, M_z)$ and (b) $(M_x, M_y, M_x^{-1}M_y^{-1})$, where M_y, M_z in (a) and M_x, M_y in (b) have no common eigenvectors, $M_{x,y,z} \in G$.
- Each finite orbit O of the induced $\bar{\Lambda}$ action (14) that consists of more than one point (i.e. each of orbits II–IV, 1–45 and Cayley orbits of size greater than one) generates a finite number of orbits of (7), which have the same size as O and correspond to different 4-tuples $(p_x, p_y, p_z, p_\infty)$ solving (11) and (12). (Recall that the parameters $\omega_{X,Y,Z,4}$ for orbits II–IV and 1–45 are specified by Lemma 39 and Table 4, while for Cayley orbits $\omega_X = \omega_Y = \omega_Z = \omega_4 = 0$.) Once a solution for $(p_x, p_y, p_z, p_\infty)$ is chosen, the orbit in $\mathcal{M}$ is completely fixed by the 7-tuple $(p_x, p_y, p_z, p_\infty, X, Y, Z)$, where (X, Y, Z) is any point in O.
- All remaining finite orbits of (7) belong to the space $\mathcal{U} \subset \mathcal{M}$ of conjugacy classes of triples of upper triangular $SL(2, \mathbb{C})$-matrices.

3. Algebraic Painlevé VI solutions

We are now prepared for the classification of PVI solutions with finite branching up to parameter equivalence.

Definition 46. Let us associate to any PVI solution branch the 7-tuple of monodromy data $(\omega_X, \omega_Y, \omega_Z, \omega_4, X, Y, Z) \in \mathbb{C}^7$ defined by (8)–(12). Two finite branch PVI solutions will be called

- equivalent if they are related by Bäcklund transformations specified in Table 1;
- parameter equivalent if their analytic continuation leads to equivalent (under $K_4 \rtimes S_3$ transformations of Subsection Section 2.2) orbits in the space of 7-tuples of monodromy data.

Remark 47. Our parameter equivalence is strictly stronger than that of [21], and is rather similar to geometric equivalence, cf. [21], Def. 8. In particular, it distinguishes solutions 3, 21 and 42 (see below), whose parameters $\boldsymbol{\theta} = (\theta_x, \theta_y, \theta_z, \theta_\infty)$ lie in the same orbit of the Okamoto affine F_4 action. Another such example is given by solutions 20 and 45.

Remark 48. In [21], p. 13 it is stated that the four-branch octahedral PVI solution [6]

$$w = \frac{(s-1)^2}{s(s-2)}, \qquad t = \frac{(s+1)(s-1)^3}{s^3(s-2)}, \tag{73}$$

with parameters $\boldsymbol{\theta} = (\vartheta, \vartheta, \vartheta, 1-3\vartheta)$ and the four-branch dihedral solution IV below are inequivalent for $\vartheta = \theta = 1/6$, although characterized by the same parameters. This seems to be incorrect; replacing $s \mapsto 1/(s+1)$ in (73) and applying to w affine D_4 transformation $s_x s_y s_z s_\delta s_x s_y s_z$, one finds solution IV with $\theta = 1/2 - 2\vartheta$. Despite the failure of the above counterexample, our parameter equivalence is presumably weaker than the equivalence under Bäcklund transformations.

Let us now examine one by one all finite orbits listed in Theorem 1 (recall that finite orbits which are not of Cayley type do not split under the action of Λ). First consider orbit I, consisting of a single point. In this case all solutions of Painlevé VI can be found explicitly. In particular, for reducible monodromy (i.e. when M_x, M_y, M_z have a common eigenvector) PVI equation linearizes and one has the following:

Proposition 49 (*Theorem 4.1 in [28]*). *All solutions of PVI corresponding to reducible monodromy are equivalent to the one-parameter family of Riccati solutions*

$$w(t) = \frac{(1+\theta_x+\theta_z - t - \theta_z t)u(t) - t(t-1)u'(t)}{(1+\theta_x+\theta_y+\theta_z)u(t)}, \tag{74}$$

realized for $\theta_\infty = -(\theta_x+\theta_y+\theta_z)$, where $u(t) = u_1(t) + \nu u_2(t)$ and $u_{1,2}(t)$ are two linear independent solutions of the following hypergeometric equation:

$$t(1-t)u'' + [(2+\theta_x+\theta_z) - (4+\theta_x+\theta_y+2\theta_z)t]u' - (2+\theta_x+\theta_y+\theta_z)(\theta_z+1)u = 0. \tag{75}$$

Remark 50. It is well-known that one-parameter family (74) contains solutions with a finite number of branches if and only if the parameters of the hypergeometric equation (75) belong to the Schwarz table, see [29] or Table 1 in [3].

The solutions of PVI in the case of "1-smaller monodromy", when one of the matrices M_x, M_y, M_z or $M_\infty = (M_z M_y M_x)^{-1}$ is equal to $\pm\mathbf{1}$, have been completely described in [30]. Any such solution is either (i) degenerate ($w = 0, 1, t, \infty$) or (ii) equivalent via Bäcklund transformations to a Riccati solution or (iii) belongs to a set of generalized Chazy solutions, expressible in terms of hypergeometric functions; see Lemma 33 in [30] for the details.

Next we consider Cayley orbits. Since in this case $\omega_X = \omega_Y = \omega_Z = \omega_4 = 0$, the 4-tuple $(p_x, p_y, p_z, p_\infty)$ can only be $(0, 0, 0, 0)$ or a permutation of $(\pm 2, \pm 2, \pm 2, \mp 2)$. This in turn implies that the 4-tuple of PVI parameters $(\theta_x, \theta_y, \theta_z, \theta_\infty)$ consists of either (i) 1 odd and 3 even integers or (ii) 1 even and 3 odd integers or (iii) all four $\theta_{x,y,z,\infty}$ have half-integer values. For $\theta_x = \theta_y = \theta_z = 0, \theta_\infty = 1$ the general solution of Painlevé VI is known:

Proposition 51. *All solutions of the sixth Painlevé equation with $\theta_x = \theta_y = \theta_z = 0, \theta_\infty = 1$ are given by Picard solutions*

$$w(t) = \wp(\nu_1 u_1 + \nu_2 u_2; u_1, u_2) + \frac{t+1}{3}, \quad \nu_{1,2} \in \mathbb{C}, 0 \le \operatorname{Re} \nu_{1,2} < 2, \tag{76}$$

where $\wp(z; u_1, u_2)$ is the Weierstrass elliptic function and $u_{1,2}(t)$ are two linearly independent solutions of the following hypergeometric equation:

$$4t(1-t)u'' + 4(1-2t)u' - u = 0, \tag{77}$$

namely,

$$u_1 = \frac{\pi}{2} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 1, t\right), \qquad u_2 = \frac{i\pi}{2} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 1, 1-t\right).$$

Proof. This statement was proved by Fuchs in [31]. □

All finite branch solutions corresponding to Cayley orbits are therefore parameter equivalent to solutions from the above two-parameter family. Equivalence under Bäcklund transformations is slightly more subtle, see e.g. [32].

There remain precisely 45 parameter inequivalent finite branch PVI solutions and three families depending on continuous parameters, which correspond to orbits 1–45 and II–IV (existence of solutions with appropriate monodromy data follows from their explicit construction below). Surprisingly, each equivalence class contains algebraic representatives that have already appeared in the literature [8,7,21–23,4,5,24,6,9,10]. Complete list of these parameter inequivalent algebraic solutions is given below. For each solution we specify the 4-tuple of PVI parameters $\boldsymbol{\theta} = (\theta_x, \theta_y, \theta_z, \theta_\infty)$, the number of branches and the explicit solution curve. We also give references to original papers where the corresponding algebraic solutions have been obtained and correct a few misprints (in solutions 13, 24, 43 and 44).

Solution II, 2 branches, $\boldsymbol{\theta} = (\theta_a, \theta_b, \theta_b, 1-\theta_a)$:

$$w(t) = \pm\sqrt{t}.$$

In Lemma 39, $X' = 2\cos 2\pi\theta_b$, $X'' = -2\cos 2\pi\theta_a$.

Solution III, 3 branches, $\boldsymbol{\theta} = (2\theta, \theta, \theta, 2/3)$:

$$w = \frac{(s-1)(s+2)}{s(s+1)}, \qquad t = \frac{(s-1)^2(s+2)}{(s+1)^2(s-2)},$$

first obtained in [4], (E.31); in the above form it appeared in [6]. In Lemma 39, $\omega = 2\cos 3\pi\theta$.

Solution IV, 4 branches, $\boldsymbol{\theta} = (\theta, \theta, \theta, 1/2)$:

$$w = \frac{s^2(s+2)}{s^2+s+1}, \qquad t = \frac{s^3(s+2)}{2s+1},$$

first obtained in [4], (E.29); in the above form it appeared in [24]. In Lemma 39, $\omega = 4\cos^2\pi\theta$.

Solution 1, 5 branches, $\boldsymbol{\theta} = (2/5, 1/5, 1/3, 2/3)$:

$$w = \frac{2(s^2+s+7)(5s-2)}{s(s+5)(4s^2-5s+10)}, \qquad t = \frac{27(5s-2)^2}{(s+5)(4s^2-5s+10)^2},$$

solution 20 in [21], p. 21.

Solution 2, 5 branches, $\boldsymbol{\theta} = (1/5, 2/5, 1/5, 2/5)$:

$$w = \frac{s^2(s-1)}{3(s-2)(s+3)}, \qquad t = \frac{2s^3(s^2-5)}{(s-2)^2(s+3)^3},$$

first found in [9], Eq. (3.3).

Solution 3, 6 branches, $\boldsymbol{\theta} = (1/2, 1/3, 1/3, 1/2)$:

$$w = -\frac{s(s+1)(s-3)^2}{3(s+3)(s-1)^2}, \qquad t = -\frac{(s+1)^3(s-3)^3}{(s-1)^3(s+3)^3},$$

first found in [8], equivalent to solution 4.1.1A; in the above form in [22], tetrahedral solution 6, p. 9.

Solution 4, 6 branches, $\boldsymbol{\theta} = (1/2, 1/4, 1/2, 2/3)$:

$$w = \frac{9s(2s^3-3s+4)}{4(s+1)(s-1)^2(2s^2+6s+1)}, \qquad t = \frac{27s^2}{4(s^2-1)^3},$$

octahedral solution 7 in [22], p. 12.

Solution 5, 6 branches, $\boldsymbol{\theta} = (1/4, 1/4, 1/3, 1/3)$:

$$w = \frac{(3s-1)(2s-1)(s+1)^3}{4s(3s^2-1)(s^2+1)}, \qquad t = \frac{(s+1)^4(2s-1)^2}{8s^3(3s^2-1)},$$

first found in [9] 3.3.3, p. 22.

Solution 6, 6 branches, $\boldsymbol{\theta} = (2/5, 1/5, 2/5, 2/3)$:

$$w = \frac{18s(s-3)}{(s-4)(s+1)(s^2+5)}, \qquad t = \frac{432s}{(s+5)(s+1)^3(s-4)^2},$$

solution 23 in [21], p. 23.

Solution 7, 6 branches, $\boldsymbol{\theta} = (1/5, 2/5, 1/5, 1/3)$:

$$w = \frac{-54s(s-7)}{(s-4)(s+1)(s^4-20s^2-35)}, \qquad t = t_6,$$

solution 22 in [21], p. 23.

Solution 8, 7 branches, $\boldsymbol{\theta} = (2/7, 2/7, 2/7, 4/7)$:

$$w = -\frac{(5s^2-8s+5)(7s^2-7s+4)}{s(s-2)(s+1)(2s-1)(4s^2-7s+7)}, \qquad t = \frac{(7s^2-7s+4)^2}{s^3(4s^2-7s+7)^2}.$$

Klein solution of [7], p. 26.

Solution 9, 8 branches, $\boldsymbol{\theta} = (1/4, 1/2, 1/4, 1/2)$:

$$w = -\frac{(s^2-2s+2)(s^2+2)^2}{4(s+1)(s^2-4s-2)(s-1)^2}, \qquad t = \frac{(s^2-2)(s^2+2)^3}{16(s+1)^3(s-1)^3},$$

first found in [9] 3.3.5, p. 23.

Solution 10, 8 branches, $\boldsymbol{\theta} = (1/3, 1/2, 1/4, 2/3)$:

$$w = \frac{s^3(2s^2-4s+3)(s^2-2s+2)}{(2s^2-2s+1)(3s^2-4s+2)}, \qquad t = \left(\frac{s^2(2s^2-4s+3)}{3s^2-4s+2}\right)^2,$$

octahedral solution 9 in [22], p. 12.

Solution 11, 8 branches, $\boldsymbol{\theta} = (1/2, 1/5, 2/5, 4/5)$:

$$w = \frac{s(s+4)(3s^4-2s^3-2s^2+8s+8)}{8(s-1)(s+1)^2(s^2+4)}, \qquad t = \frac{s^5(s+4)^3}{4(s-1)(s+1)^3(s^2+4)^2},$$

solution 24 in [21], p. 21.

Solution 12, 8 branches, $\boldsymbol{\theta} = (2/5, 1/2, 2/5, 4/5)$:

$$w = \frac{s^2(s+4)^2(5s^3+2s^2-4s-8)}{4(s-1)(s+1)^2(s^2+4)(s^2+3s+6)}, \qquad t = t_{11},$$

solution 25 in [21], p. 21.

Solution 13, 9 branches, $\boldsymbol{\theta} = (2/5, 2/5, 2/5, 2/3)$:

$$w = \frac{1}{2} + \frac{350s^3+63s^2-6s-2}{30s(2s+1)u},$$
$$t = \frac{1}{2} + \frac{(25s^4+170s^3+42s^2+8s-2)u}{54s^3(5s+4)^2},$$
$$u^2 = s(8s+1)(5s+4),$$

solution 27 in [21], p. 23 (parameters in [21] are defined with a misprint, which is corrected by interchanging $\theta_3 \leftrightarrow \theta_4$).

Solution 14, 9 branches, $\boldsymbol{\theta} = (1/5, 1/5, 1/5, 1/3)$:

$$w = \frac{1}{2} - \frac{(s-1)\big(5(s^6+1)+58(s^5+s)+1771(s^4+s^2)+8620s^3\big)u}{8s(s+1)(5s^3+25s^2+95s+3)(3s^3+95s^2+25s+5)},$$
$$t = \frac{1}{2} - \frac{(s-1)\big(25(s^8+1)+760(s^7+s)+4924(s^6+s^2)+75464(s^5+s^3)+329174s^4\big)}{2048s(s+1)^5u},$$
$$u^2 = s(5s^2+118s+5),$$

first found in [9], p. 11.

Solution 15, 10 branches, $\boldsymbol{\theta} = (1/2, 1/5, 1/2, 3/5)$:

$$w = \frac{(s^2-5)(s^2+5)(s^5+5s^4-20s^3+75s+75)}{(s+1)^2(s+5)(s^2-4s+5)(s^4+6s^2-75)},$$

$$t = \frac{2(s^2+5)^3(s^2-5)^2}{(s+1)^3(s+5)^3(s^2-4s+5)^2},$$

solution 28 in [21], p. 21.

Solution 16, 10 branches, $\boldsymbol{\theta} = (0, 0, 0, -4/5)$:

$$w = \frac{(s-1)^2(3s+1)^2(s^2+4s-1)(119s^8-588s^6+314s^4-108s^2+7)^2}{(s+1)^3(3s-1)P(s)},$$

$$t = \frac{(s-1)^5(3s+1)^3(s^2+4s-1)}{(s+1)^5(3s-1)^3(s^2-4s-1)},$$

$$\begin{aligned} P(s) = {} & 42483s^{18} - 719271s^{16} + 5963724s^{14} + 13758708s^{12} - 7616646s^{10} + 1642878s^8 - 259044s^6 \\ & + 34308s^4 - 2133s^2 + 49, \end{aligned}$$

first obtained in [4], the above parametrization corresponds to icosahedron solution (H_3) in [5], p. 76.

Solution 17, 10 branches, $\boldsymbol{\theta} = (0, 0, 0, -2/5)$:

$$w = \frac{(s-1)^4(3s+1)^2(s^2+4s-1)(11s^4-30s^2+3)^2}{(s+1)(3s-1)(3s^2+1)P(s)},$$

$$t = \frac{(s-1)^5(3s+1)^3(s^2+4s-1)}{(s+1)^5(3s-1)^3(s^2-4s-1)},$$

$$P(s) = 121s^{12} - 1942s^{10} + 63015s^8 - 28852s^6 + 4855s^4 - 342s^2 + 9,$$

great icosahedron solution $(H_3)'$ in [5], p. 77.

Solution 18, 10 branches, $\boldsymbol{\theta} = (1/3, 1/3, 1/3, 4/5)$:

$$w = \frac{s^2(s+2)(s^2+1)(2s^2+3s+3)}{2(s^2+s+1)(3s^2+3s+2)}, \qquad t = \frac{s^5(s+2)(2s^2+3s+3)^2}{(2s+1)(3s^2+3s+2)^2},$$

solution 29 in [21], p. 23.

Solution 19, 10 branches, $\boldsymbol{\theta} = (1/3, 1/3, 1/3, 2/5)$:

$$w = \frac{s^4(s+2)(2s^2+3s+3)(7s^2+10s+7)}{(3s^2+3s+2)\big(4(s^6+1)+12(s^5+s)+15(s^4+s^2)+10s^3\big)}, \qquad t = t_{18},$$

solution 30 in [21], p. 23.

Solution 20, 12 branches, $\boldsymbol{\theta} = (1/2, 1/2, 1/2, 2/3)$:

$$w = \frac{1}{2} + \frac{45s^6+20s^5+95s^4+92s^3+39s^2-3}{4(5s^2+1)(s+1)^2u},$$

$$t = \frac{1}{2} + \frac{s(2s+1)^2(27s^4+28s^3+26s^2+12s+3)}{(s+1)^3u^3},$$

$$u^2 = (2s+1)(9s^2+2s+1),$$

octahedral solution 12 in [22], p. 13.

Solution 21, 12 branches, $\boldsymbol{\theta} = (1/3, 1/2, 1/2, 2/3)$:

$$w = \frac{4(s+1)(3s^2-4s+2)(7s^4+16s^3+4s^2-4)}{s^3(s-2)(s^2+4s+6)(s^4-4s^2+32s-28)},$$

$$t = \frac{16(s+1)^4(3s^2-4s+2)^2}{s^4(s-2)^4(s^2+4s+6)^2},$$

octahedral solution 11 in [22], p. 12.

Solution 22, 12 branches, $\boldsymbol{\theta} = (1/3, 1/3, 1/5, 2/5)$:

$$w = \frac{1}{2} + \frac{140s^6 + 1029s^5 - 1023s^4 + 360s^3 - 288s^2 + 27s + 27}{18u(s+1)(7s^3 - 3s^2 - s + 1)},$$
$$t = \frac{1}{2} + \frac{40s^6 + 540s^5 - 765s^4 + 540s^3 - 270s^2 + 27}{6u(8s^2 - 9s + 3)(s+1)^2},$$
$$u^2 = 3(5s+1)(8s^2 - 9s + 3),$$

solution 36 in [21], p. 22.

Solution 23, 12 branches, $\boldsymbol{\theta} = (1/5, 1/5, 1/3, 1/2)$:

$$w = \frac{1}{2} + \frac{(3s+5)(8s^4 - 10s^3 + 12s^2 - 13s + 11)}{2(2s^3 - 15s + 5)u},$$
$$t = \frac{1}{2} - \frac{8s^6 + 20s^3 - 15s^2 + 66s - 15}{2(8s^2 - 5s + 5)u},$$
$$u^2 = (3s+5)(8s^2 - 5s + 5),$$

solution 34 in [21], p. 21.

Solution 24, 12 branches, $\boldsymbol{\theta} = (2/5, 2/5, 1/3, 1/2)$:

$$w = \frac{1}{2} - \frac{(3s+5)(16s^5 - 8s^4 + 18s^3 - 8s^2 + 115s + 3)}{2(26s^3 + 60s^2 + 15s + 35)u},$$
$$t = t_{23}, \qquad u = u_{23},$$

solution 35 in [21], p. 22 (in [21] there is a sign misprint in the formula for w).

Solution 25, 12 branches, $\boldsymbol{\theta} = (2/5, 1/3, 1/2, 4/5)$:

$$w = -\frac{9s(s^2+1)(3s-4)(15s^4 - 5s^3 + 3s^2 - 3s + 2)}{(2s-1)^2(9s^2+4)(9s^2+3s+10)},$$
$$t = \frac{27s^5(s^2+1)^2(3s-4)^3}{4(2s-1)^3(9s^2+4)^2},$$

solution 33 (generic icosahedral solution) in [21], Th. B, p. 4.

Solution 26, 15 branches, $\boldsymbol{\theta} = (1/3, 1/3, 1/3, 3/5)$:

$$w = \frac{1}{2} - \frac{250s^6 + 500s^5 + 518s^4 + 261s^3 + 76s^2 + 13s + 2}{2(s+2)(5s+1)(5s^3 + 6s^2 + 3s + 1)u},$$
$$t = \frac{1}{2} - \frac{3(500s^7 + 925s^6 + 1164s^5 + 830s^4 + 340s^3 + 105s^2 + 20s + 4)}{2(s+2)^2(5s+1)u^3},$$
$$u^2 = (4s^2 + s + 1)(5s+1),$$

solution 38 in [21], p. 26.

Solution 27, 15 branches, $\boldsymbol{\theta} = (1/3, 1/3, 1/3, 1/5)$:

$$w = \frac{1}{2} - \frac{1000s^8 + 2425s^7 + 4171s^6 + 3805s^5 + 1999s^4 + 874s^3 + 244s^2 + 58s + 4}{4(s+2)(25s^6 + 135s^5 + 111s^4 + 91s^3 + 36s^2 + 6s + 1)u},$$
$$t = t_{26}, \qquad u = u_{26},$$

solution 37 in [21], p. 26.

Solution 28, 15 branches, $\boldsymbol{\theta} = (3/5, 3/5, 2/3, 2/3)$:

$$w = \frac{1}{2} - \frac{2s^9 + 20s^8 + 53s^7 - 89s^6 - 605s^5 - 851s^4 - 1389s^3 - 5775s^2 - 10125s - 5625}{2(s^2-5)(s^2-6s-15)(s^2+4s+5)u},$$
$$t = \frac{1}{2} - \frac{(2s^7 + 10s^6 - 90s^4 - 135s^3 + 297s^2 + 945s + 675)u}{18(4s^2 + 15s + 15)^2(s^2-5)},$$
$$u^2 = 3(s+5)(4s^2 + 15s + 15),$$

solution 40 in [21], p. 22.

Solution 29, 15 branches, $\boldsymbol{\theta} = (1/3, 1/3, 4/5, 4/5)$:

$$w = \frac{1}{2} + \frac{14s^5 + 61s^4 - 66s^3 - 660s^2 - 900s - 225}{6(s+1)(s^2-5)u},$$
$$t = t_{28}, \qquad u = u_{28},$$

solution 39 in [21], p. 22.

Solution 30, 16 branches, $\boldsymbol{\theta} = (1/2, 1/2, 1/2, 3/4)$:

$$w = -\frac{(1+i)(s^2-1)(s^2+2is+1)(s^2-2is+1)^2 P(s)}{4s(s^2+i)(s^2-i)^2(s^2+(1+i)s-i)Q(s)},$$
$$t = \frac{(s^2-1)^2(s^4+6s^2+1)^3}{32s^2(s^4+1)^3},$$
$$P(s) = s^8 - (2-2i)s^7 - (6+2i)s^6 + (10+2i)s^5 + 4is^4 + (10-2i)s^3 + (6-2i)s^2 - (2+2i)s - 1,$$
$$Q(s) = s^6 - (3+3i)s^5 + 3is^4 + (4-4i)s^3 + 3s^2 + (3+3i)s + i,$$

octahedral solution 13 in [22], p. 13.

Solution 31, 18 branches, $\boldsymbol{\theta} = (1/3, 1/3, 1/3, 1/3)$:

$$w = \frac{1}{2} - \frac{8s^7 - 28s^6 + 75s^5 + 31s^4 - 269s^3 + 318s^2 - 166s + 56}{18u(s-1)(3s^3 - 4s^2 + 4s + 2)},$$
$$t = \frac{1}{2} + \frac{(s+1)\big(32(s^8+1) - 320(s^7+s) + 1112(s^6+s^2) - 2420(s^5+s^3) + 3167s^4\big)}{54u^3 s(s-1)},$$
$$u^2 = s(8s^2 - 11s + 8).$$

A solution with equivalent parameters was first obtained in [5] (great dodecahedron solution $(H_3)''$, see pp. 78–87 in the preprint version of [5] for the explicit form), the above elliptic parametrization was produced in [21], Th. C, p. 4.

Solution 32, 18 branches, $\boldsymbol{\theta} = (4/7, 4/7, 4/7, 1/3)$:

$$w = \frac{1}{2} - \frac{P(s)u}{Q(s)}, \qquad t = \frac{1}{2} - \frac{R(s)u}{432s(s+1)^2(s^2+s+7)^2}, \qquad u^2 = s(s^2+s+7),$$
$$P(s) = s^{10} + 5s^9 + 24s^8 + 20s^7 - 266s^6 - 2874s^5 - 14812s^4 - 40316s^3 - 85359s^2 - 100067s - 67396,$$
$$Q(s) = 16(s+1)(s^2+s+7)(5s^6 + 63s^5 + 252s^4 + 854s^3 + 1449s^2 + 1827s + 2030),$$
$$R(s) = s^9 - 84s^6 - 378s^5 - 1512s^4 - 5208s^3 - 7236s^2 - 8127s - 784,$$

first appeared in [22], p. 22.

Solution 33, 18 branches, $\boldsymbol{\theta} = (1/3, 1/7/, 1/7, 6/7)$:

$$w = 1 + \frac{(3s-2)(s^2-2s+4)^2}{4(s+2)(s-1)^2(s^2-s+1)(3s^2-4s+4)} \times \frac{-14s^5 + 25s^4 - 20s^3 - 8s^2 + 16s - 8 - 8(s-1)(s^2-s+1)u}{(2s+1)(3s^3 - 10s^2 + 6s - 2) - 14(s-1)u},$$
$$t = \frac{1}{2} - \frac{14s^9 - 105s^8 + 252s^7 - 392s^6 + 420s^5 - 336s^4 + 112s^3 + 72s^2 - 96s + 32}{16(s+2)^2(s-1)^3(s^2-s+1)u},$$
$$u^2 = (2s+1)(1-s)(s^2-s+1),$$

solution (3.16)–(3.17) in [10], p. 15.

Solution 34, 18 branches, $\boldsymbol{\theta} = (2/7, 2/7, 2/7, 1/3)$:

$$w = \frac{1}{2} - \frac{(3s^8 - 2s^7 - 4s^6 - 204s^5 - 536s^4 - 1738s^3 - 5064s^2 - 4808s - 3199)u}{4(s+1)(s^2+s+7)(s^6 + 196s^3 + 189s^2 + 756s + 154)},$$
$$t = t_{32}, \quad u = u_{32},$$

first appeared in [22], p. 17, Eq. (12).

Solution 35, 20 branches, $\boldsymbol{\theta} = (0, 0, 1/10, 9/10)$:

$$w = \frac{1}{2} - \frac{9s^5 - 49s^4 - 822s^3 + 238s^2 - 1699s + 1299}{2(3s-7)(s^2-2s+17)u},$$

$$t = \frac{1}{2} - \frac{P(s)}{Q(s)u^3}, \qquad u^2 = (9s^2 - 2s + 9)(s^2 - 2s + 17)$$

$$P(s) = 27s^{10} - 630s^9 + 4055s^8 + 30520s^7 - 174970s^6 + 258492s^5 - 724490s^4 + 600760s^3 - 1097825s^2 + 186570s - 131085,$$

$$Q(s) = 2(s^2 - 2s + 17)(s^2 - 18s + 1),$$

solution 45 of [21], first obtained explicitly in [23], p. 7.

Solution 36, 20 branches, $\boldsymbol{\theta} = (0, 0, 3/10, 7/10)$:

$$w = \frac{1}{2} - \frac{(s+3)(9s^4 - 100s^3 + 118s^2 - 228s - 55)}{(6s^3 - 42s^2 - 30s - 62)u},$$

$$t = t_{35}, \quad u = u_{35},$$

solution 44 of [21], first obtained explicitly in [23], p. 8.

Solution 37, 20 branches, $\boldsymbol{\theta} = (1/3, 1/3, 1/2, 2/5)$:

$$w = \frac{1}{2} + \frac{(s+3)P(s)}{18(s^2+1)(s^6 - 7s^4 + 42s^3 - 45s^2 + 34s + 7)u},$$

$$t = \frac{1}{2} - \frac{(s+3)Q(s)}{2(s^2+1)^2u^3}, \qquad u^2 = 3(s+3)(8s^2 - 13s + 17),$$

$$P(s) = 28s^9 - 235s^8 + 556s^7 - 1334s^6 + 2174s^5 - 3854s^4 + 4360s^3 - 4738s^2 + 2362s - 1047,$$

$$Q(s) = 8s^{10} + 100s^7 - 135s^6 + 834s^5 - 1205s^4 + 2280s^3 - 1365s^2 + 890s + 321,$$

solution 43 in [21], p. 24.

Solution 38, 20 branches, $\boldsymbol{\theta} = (1/3, 1/3, 1/2, 4/5)$:

$$w = \frac{1}{2} + \frac{(s+3)(8s^6 - 28s^5 + 85s^4 - 196s^3 + 214s^2 - 196s + 41)}{6(s^2+1)(3s^2 - 4s + 5)u},$$

$$t = t_{37}, \qquad u = u_{37},$$

solution 42 in [21], p. 24.

Solution 39, 24 branches, $\boldsymbol{\theta} = (1/3, 1/3, 1/3, 1/2)$:

$$w = \frac{1}{2} - \frac{P(s)}{R(s)u}, \qquad t = \frac{1}{2} + \frac{(s^2+4s-2)Q(s)}{2(s+2)(3s^2-2s+2)^2u^3}, \qquad u^2 = (8s^2 - 7s + 2)(s+2),$$

$$P(s) = 16s^{11} + 72s^{10} + 50s^9 - 242s^8 - 3143s^7 + 6562s^6 - 8312s^5 + 9760s^4 - 9836s^3 + 6216s^2 - 2288s + 416,$$

$$Q(s) = 8s^{10} + 16s^9 + 24s^8 - 84s^7 + 429s^6 - 312s^5 + 258s^4 - 288s^3 + 288s^2 - 128s + 32,$$

$$R(s) = 2(3s^2 - 2s + 2)(26s^6 + 18s^5 - 75s^4 + 50s^3 + 270s^2 - 312s + 104),$$

solution 46 in [21], p. 27.

Solution 40, 30 branches, $\boldsymbol{\theta} = (1/15, 1/15, 7/30, 23/30)$:

$$w = \frac{1}{2} + \frac{(s+1)(s^8 + 8s^7 + 90s^6 + 348s^5 + 972s^4 + 1296s^3 + 4374s^2 + 8748s + 19683)}{2(s+3)^2(s^4 - 4s^3 - 6s^2 + 81)u},$$

$$t = \frac{1}{2} + \frac{(s+1)^2(s+9)^2P(s)}{2(s-3)^2(s+3)^5(s^2+9)u^3}, \qquad u^2 = (s+1)(s+9)(s^2+9)(s^2+4s+9),$$

$$P(s) = s^{14} + 10s^{13} + 63s^{12} + 180s^{11} + 621s^{10} + 3942s^9 + 26595s^8 + 99576s^7 + 239355s^6 + 319302s^5 + 452709s^4 + 1180980s^3 + 3720087s^2 + 5314410s + 4782969,$$

solution 47 of [21], first obtained explicitly in [23], p. 9.

Solution 41, 30 branches, $\boldsymbol{\theta} = (2/15, 2/15, 1/30, 29/30)$:

$$w = \frac{1}{2} + \frac{(s+9)Q(s)}{2(s-3)(s+3)^4(s^2+9)u}, \qquad t = t_{40}, \qquad u = u_{40},$$
$$Q(s) = s^9 + 7s^8 + 36s^7 + 36s^6 + 126s^5 + 1170s^4 + 8100s^3 + 18468s^2 + 24057s - 6561,$$

solution 48 of [21], first obtained explicitly in [23], p. 9.

Solution 42, 36 branches, $\boldsymbol{\theta} = (0, 0, 1/6, 5/6)$:

$$w = \frac{1}{2} - \frac{4s^9 - 24s^8 + 84s^7 - 240s^6 + 96s^5 + 1401s^4 - 6396s^3 + 11136s^2 - 8160s - 401}{2(2s^2-2s+5)(s^3-3s^2+3s-11)u},$$
$$t = \frac{1}{2} - \frac{(s-2)(s+4)P(s)}{4(s^2-7s+1)(s^2-4s+13)(2s^2-2s+5)u^3},$$
$$u^2 = (s^2-4s+13)(2s^2-2s+5)(2s^4+2s^3-3s^2-58s+107),$$
$$\begin{aligned} P(s) \;=\; & 32s^{16} - 640s^{15} + 6432s^{14} - 46016s^{13} + 266968s^{12} - 1228152s^{11} + 4546772s^{10} \\ & - 13723024s^9 + 34628427s^8 - 74456536s^7 + 139564088s^6 - 224784264s^5 \\ & + 300342142s^4 - 299494736s^3 + 197723868s^2 - 68764168s + 17918807, \end{aligned}$$

solution 49 of [21], first obtained explicitly in [23], p. 10.

Solution 43, 40 branches, $\boldsymbol{\theta} = (3/20, 3/20, 3/20, 17/20)$:

$$w = \frac{1}{2} + \frac{(s^2-18s+1)(s^2-2s+17)\,(u_{35})^2 + 8(s+1)(3s^3-21s^2-15s-31)uv}{32(s^3+57s^2-69s+75)(s^2-1)v},$$
$$t = \frac{1}{2} + \frac{P_{35}(s)u}{1024(s-9)^2(s^2-1)^3(5s^2-2s+13)},$$
$$u^2 = 2(s-9)(s^2-1), \qquad v^2 = -(s-1)(s-9)(5s^2-2s+13),$$

solution 50 of [21], first obtained explicitly in [23], p. 9. (The formula (6) for v in [23], p. 8 is incorrect and should be replaced with $v^2 = -2(j+1)(5j^2-2j+13)$. This is undoubtedly a typing error, because the Maple file accompanying Arxiv version of [23] contains correct expressions, which yield a solution equivalent to the above.)

Solution 44, 40 branches, $\boldsymbol{\theta} = (1/20, 1/20, 1/20, 19/20)$:

$$w = \frac{1}{2} + \frac{(s^2-18s+1)\,(u_{35})^2 + 4(s-1)(3s-7)uv}{64(s+3)(s+1)^2 v},$$
$$t = t_{43}, \qquad u = u_{43}, \qquad v = v_{43},$$

solution 51 of [21], in explicit form first obtained in [23], p. 8 (with the same misprints as solution 43).

Solution 45, 72 branches, $\boldsymbol{\theta} = (1/12, 1/12, 1/12, 11/12)$:

$$w = \frac{1}{2} + \frac{2(s^2-4s+13)(s^2-7s+1)\,(u_{42})^2 + 9(s-1)(s^3+27s^2-57s+79)uv}{6(2s-7)^2(s^2-1)(2s^2+s+17)(s^3-3s^2+3s-11)v},$$
$$t = \frac{1}{2} + \frac{(s-2)(s+4)P_{42}(s)}{54(2s-7)(s^2-1)(s^2-2s+6)u^3},$$
$$u^2 = (2s-7)(s^2-1)(2s^2+s+17)(4s^2-13s+19),$$
$$v^2 = -(s+1)(s^2-2s+6)(4s^2-13s+19),$$

solution 52 of [21], in explicit form first obtained in [23], pp. 10–11.

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References

[1] E.L. Ince, Ordinary Differential Equations, Dover, New York, 1956.
[2] H. Watanabe, Birational canonical transformations and classical solutions of the sixth Painlevé equation, Ann. Sc. Norm. Super Pisa Cl. Sci. 27 (1999) 379–425.
[3] P. Boalch, Towards a nonlinear Schwarz's list, Preprint math.CA/0707.3375.

[4] B. Dubrovin, Geometry of 2D topological field theories, in: Lect. Notes Math., vol. 1620, Springer, 1995, pp. 120–348. Preprint hep-th/9407018.
[5] B. Dubrovin, M. Mazzocco, Monodromy of certain Painlevé VI transcendents and reflection groups, Invent. Math. 141 (2000) 55–147. Preprint math.AG/9806056.
[6] N.J. Hitchin, A lecture on octahedron, Bull. Lond. Math. Soc. 35 (2003) 577–600.
[7] P. Boalch, From Klein to Painleve via Fourier, Laplace and Jimbo, Proc. Lond. Math. Soc. 90 (2005) 167–208. Preprint math.AG/0308221.
[8] F.V. Andreev, A.V. Kitaev, Transformations $RS_4^2(3)$ of the ranks ≤ 4 and algebraic solutions of the sixth Painleve equation, Comm. Math. Phys. 228 (2002) 151–176. Preprint nlin.SI/0107074.
[9] A.V. Kitaev, Grothendieck's dessins d'enfants, their deformations, and algebraic solutions of the sixth Painlevé and Gauss hypergeometric equations, Algebra i Analiz 17 (2005) 224–275. Preprint nlin.SI/0309078.
[10] A.V. Kitaev, Remarks towards a classification of $RS_4^2(3)$-transformations and algebraic solutions of the sixth Painlevé equation, in: Sémin. Congr., vol. 14, Soc. Math., France, 2006, pp. 199–227. Preprint math.CA/0503082.
[11] A. Borodin, P. Deift, Fredholm determinants, Jimbo–Miwa–Ueno tau-functions, and representation theory, Comm. Pure Appl. Math. 55 (2002) 1160–1230. Preprint math-ph/0111007.
[12] O. Lisovyy, On Painlevé VI transcendents related to the Dirac operator on the hyperbolic disk, J. Math. Phys. 49 (2008) 093507. Preprint arXiv:0710.5744 [math-ph].
[13] K. Iwasaki, A modular group action on cubic surfaces and the monodromy of the Painlevé VI equation, Proc. Japan Acad. Ser. A 78 (2002) 131–135.
[14] M. Jimbo, Monodromy problem and the boundary condition for some Painlevé equations, Publ. RIMS, Kyoto Univ. 18 (1982) 1137–1161.
[15] S. Cantat, F. Loray, Holomorphic dynamics, Painlevé VI equation and character varieties, Preprint arxiv:0711.1579 [math.DS].
[16] K. Iwasaki, T. Uehara, An ergodic study of Painleve VI, Math. Ann. 338 (2007) 295–345. Preprint arXiv:math/0604582 [math.AG].
[17] K. Iwasaki, On algebraic solutions to Painleve VI, Preprint arXiv:0809.1482 [math.CA].
[18] A.V. Kitaev, Quadratic transformations for the sixth Painlevé equation, Lett. Math. Phys. 21 (1991) 105–111.
[19] A.V. Kitaev, R. Vidunas, Quadratic transformations of the sixth Painlevé equation with application to algebraic solutions, Math. Nachr. 280 (2007) 1834–1855. Preprint math.CA/0511149.
[20] K. Iwasaki, Finite branch solutions to Painlevé VI around a fixed singular point, Adv. Math. 217 (2008) 1889–1934. Preprint math.AG/0704.0679.
[21] P. Boalch, The fifty-two icosahedral solutions to Painlevé VI, J. Reine Angew. Math. 596 (2006) 183–214. Preprint math.AG/0406281.
[22] P. Boalch, Some explicit solutions to the Riemann–Hilbert problem, in: IRMA Lect. Math. Theor. Phys., 9, Eur. Math. Soc., Zurich, 2007, pp. 85–112. Preprint math.DG/0501464.
[23] P. Boalch, Higher genus icosahedral Painlevé curves, Funkcial. Ekvac. 50 (2007) 19–32. Preprint math.AG/0506407.
[24] N.J. Hitchin, Poncelet polygons and the Painlevé transcendents, in: Geometry and Analysis, Tata Inst. Fund. Res, Bombay, 1995, pp. 151–185.
[25] M. Noumi, Y. Yamada, A new Lax pair for the sixth Painlevé equation associated with $\widehat{\mathfrak{so}}(8)$, in: T. Kawai, K. Fujita (Eds.), Microlocal Analysis and Complex Fourier Analysis, World Scientific, 2002, Preprint math-ph/0203029.
[26] W.J.R. Crosby, Problems and solutions, Amer. Math. Monthly 53 (1946) 104–107.
[27] L. Kronecker, Les facteurs irréductibles de l'expression $x^n - 1$, J. Math. Pures Appl. 19 (1854) 177–192.
[28] M. Mazzocco, Rational solutions of the Painlevé VI equation, J. Phys. A34 (2001) 2281–2294. Preprint nlin.SI/0007036.
[29] H.A. Schwarz, Über diejenigen Fälle, in welchen die Gaussische hypergeometrische reihe eine algebraische Funktion ihres vierten Elements darstellt, J. Reine Angew. Math. 75 (1873) 292–335.
[30] M. Mazzocco, The geometry of the classical solutions of the Garnier systems, Int. Math. Res. Not. IMRN 12 (2002) 613–646. Preprint math.CA/0106208.
[31] R. Fuchs, Über lineare homogene Differentialgleichungen zweiter Ordnung mit im drei im Endrichen gelegene wesentlich singulären Stellen, Math. Ann. 63 (1907) 301–321.
[32] M. Mazzocco, Picard and Chazy solutions to the Painleve VI equation, Math. Ann. 321 (2001) 157–195. Preprint math.AG/9901054.